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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird
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- Same idea as Flajolet-Martin algorithm and HyperLogLog, except they use discrete hash functions.
\( s \) is the minimum of \( d \) points chosen uniformly at random on \([0, 1]\). Where \( d = \# \) distinct elements.

\[
E[s] = \frac{1}{d+1}
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\( \hat{d} \) output by the algorithm is correct if \( s \) exactly equals its expectation.

Does this mean \( E[\hat{d}] = d \)? No, but:

Approximation is robust: if

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|s - E[s]| \leq \epsilon \cdot E[s] \quad \text{for any} \quad \epsilon \in (0, \frac{1}{2})
\]

\( (1 - 4\epsilon) d \leq \hat{d} \leq (1 + 4\epsilon) d \)

\[ 5 \]
**PERFORMANCE IN EXPECTATION**

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How should we set \( k \) if we want \( 4\epsilon \cdot d \) error with probability \( \geq 1 - \delta \)?

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**Chebyshev Inequality:**

\[ \Pr \left[ \left| d - \hat{d} \right| \geq 4\epsilon \cdot d \right] \leq \frac{\text{Var}[s]}{\left( \epsilon \mathbb{E}[s] \right)^2} = \frac{\mathbb{E}[s]^2}{\epsilon^2 \mathbb{E}[s]^2} = \frac{1}{k \cdot \epsilon^2} \]

How should we set \( k \) if we want \( 4\epsilon \cdot d \) error with probability \( \geq 1 - \delta \)?

\[ k = \frac{1}{\epsilon^2 \cdot \delta}. \]

\( s_j \): minimum of \( d \) distinct hashes chosen randomly over \([0, 1]\). \( s = \frac{1}{k} \sum_{j=1}^{k} s_j \).

\( \hat{d} = \frac{1}{s} - 1 \): estimate of \# distinct elements \( d \).
\[ \mathbf{s} = \frac{1}{k} \sum_{j=1}^{k} s_j. \] Have already shown that for \( j = 1, \ldots, k \):

\[
\mathbb{E}[s_j] = \frac{1}{d + 1} \quad \implies \quad \mathbb{E}[s] = \frac{1}{d + 1} \quad \text{(linearity of expectation)}
\]

\[
\text{Var}[s_j] \leq \frac{1}{(d + 1)^2} \quad \implies \quad \text{Var}[s] \leq \frac{1}{k \cdot (d + 1)^2} \quad \text{(linearity of variance)}
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**Chebyshev Inequality:**

\[
\Pr \left[ \left| d - \hat{d} \right| \geq 4\epsilon \cdot d \right] \leq \frac{\text{Var}[s]}{(\epsilon \mathbb{E}[s])^2} = \frac{\mathbb{E}[s]^2}{(\epsilon^2 \mathbb{E}[s]^2)} = \frac{1}{k \cdot (d + 1)^2} = \frac{\epsilon^2 \cdot \delta}{\epsilon^2} = \delta.
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HASHING FOR DISTINCT ELEMENTS:

• Let $h_1, h_2, \ldots, h_k : U \rightarrow [0, 1]$ be random hash functions

• $s_1, s_2, \ldots, s_k := 1$

• For $i = 1, \ldots, n$
  • For $j=1, \ldots, k$, $s_j := \min(s_j, h_j(x_i))$

• $s := \frac{1}{k} \sum_{j=1}^{k} s_j$

• Return $\hat{d} = \frac{1}{s} - 1$

• Setting $k = \frac{1}{\epsilon^2 \delta}$, algorithm returns $\hat{d}$ with $|d - \hat{d}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$. 
Space Complexity

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- \( \delta = 5\% \) failure rate gives a factor 20 overhead in space complexity.
IMPROVED FAILURE RATE

How can we improve our dependence on the failure rate $\delta$?
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**The median trick:** Run $t = O(\log 1/\delta)$ trials each with failure probability $\delta' = 1/4$ – each using $k = \frac{1}{\delta' \epsilon^2} = \frac{4}{\epsilon^2}$ hash functions.
How can we improve our dependence on the failure rate $\delta$?

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  $$\hat{d} = median(\hat{d}_1, \ldots, \hat{d}_t).$$

- If $> 1/2$ of trials fall in $[(1 - 4\epsilon)d, (1 + 4\epsilon)d]$, then the median will.
THE MEDIAN TRICK

• $\hat{d}_1, \ldots, \hat{d}_t$ are the outcomes of the $t$ trials, each falling in
  
  $$[(1 - 4\epsilon)d, (1 + 4\epsilon)d]$$

  with probability at least 3/4. Let $\hat{d} = median(\hat{d}_1, \ldots, \hat{d}_t)$.

  What is the probability that the median $\hat{d}$ falls in $[(1 - 4\epsilon)d, (1 + 4\epsilon)d]$?
• \(\hat{d}_1, \ldots, \hat{d}_t\) are the outcomes of the \(t\) trials, each falling in 
\[\left[(1 - 4\epsilon)d, (1 + 4\epsilon)d\right]\]
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What is the probability that the median \(\hat{d}\) falls in \([ (1 - 4\epsilon)d, (1 + 4\epsilon)d]\)?

• Let \(X\) be the \# of trials falling in \([ (1 - 4\epsilon)d, (1 + 4\epsilon)d]\).
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\[
Pr\left(\hat{d} \not\in [(1 - 4\epsilon)d, (1 + 4\epsilon)d]\right) \leq Pr\left(X \leq \frac{1}{2} \cdot t\right)
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Apply Chernoff bound:
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Apply Chernoff bound:

\[
\Pr \left( |X - \mathbb{E}[X]| \geq \frac{1}{3} \mathbb{E}[X] \right) \leq 2 \exp \left( - \frac{\frac{12}{3} \cdot \frac{3}{4} t}{2 + 1/3} \right) = e^{-\Theta(t)}.
\]
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  Apply Chernoff bound:

  $$\Pr \left( |X - \mathbb{E}[X]| \geq \frac{1}{3} \mathbb{E}[X] \right) \leq 2 \exp \left( -\frac{1^2 \cdot \frac{3}{4} t}{2 + 1/3} \right) = e^{-\Theta(t)}.$$  

• Setting $t = O(\log(1/\delta))$ gives failure probability $e^{-\log(1/\delta)} = \delta$.  

Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns

$$\hat{d} \in [(1 - 4\epsilon)d, (1 + 4\epsilon)d]$$

with probability at least $1 - \delta$. 
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**Total Space Complexity:** $t$ trials, each using $k = \frac{1}{\epsilon^2 \delta'}$ hash functions, for $\delta' = 1/4$. Space is $\frac{4t}{\epsilon^2} = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$ real numbers (the minimum value of each hash function).
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No dependence on the number of distinct elements \( d \) or the number of items in the stream \( n! \). Both can be very large.
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No dependence on the number of distinct elements \( d \) or the number of items in the stream \( n \)! Both can be very large.

**A note on the median:** The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).
Our algorithm uses continuous valued fully random hash functions.
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- The idea of using the minimum hash value of $x_1, \ldots, x_n$ to estimate the number of distinct elements naturally extends to when the hash functions map to discrete values.
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- The idea of using the minimum hash value of $x_1, \ldots, x_n$ to estimate the number of distinct elements naturally extends to when the hash functions map to discrete values.
- Flajolet-Martin (LogLog) algorithm and HyperLogLog.
DISTINCT ELEMENTS IN PRACTICE

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Distinct Elements in Practice

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Estimate $\#$ distinct elements based on maximum number of trailing zeros $m$. 
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Estimate $\#$ distinct elements based on maximum number of trailing zeros $m$.
The more distinct hashes we see, the higher we expect this maximum to be.
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Estimate \# distinct elements based on maximum number of trailing zeros \( m \).

\[
\Pr(h(x) \text{ has } x \text{ trailing zeros}) = \frac{1}{2^x}.
\]

So with \( d \) distinct hashes, expect to see 1 with \( \log d \) trailing zeros. Expect \( m \approx \log \log d \).

\( m \) takes \( \log \log d \) bits to store.

Total Space: \( O(\log \log d \epsilon + \log d) \) for an \( \epsilon \) approximate count.

Note: Careful averaging of estimates from multiple hash functions.
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Estimate $\#$ distinct elements based on maximum number of trailing zeros $m$.

With $d$ distinct elements, roughly what do we expect $m$ to be?

- a) $O(1)$  
- b) $O(\log d)$  
- c) $O(\sqrt{d})$  
- d) $O(d)$
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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<tr>
<td>$h(x_n)$</td>
<td>1011000</td>
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With $d$ distinct elements, roughly what do we expect $m$ to be?

$$\Pr(h(x_i) \text{ has } \log d \text{ trailing zeros}) = \frac{1}{2^{\log d}}$$
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

<table>
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Estimate \# distinct elements based on maximum number of trailing zeros \( m \).

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LOGLOG COUNTING OF DISTINCT ELEMENTS

Flajolet-Martin (LogLog) algorithm and HyperLogLog.

| $h(x_1)$ | 1010010 |
| $h(x_2)$ | 1001100 |
| $h(x_3)$ | 1001110 | Estimate $\#$ distinct elements based on maximum number of trailing zeros $m$.
| $\vdots$ | $\vdots$ |
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- Given data structures (sketches) \(HLL(x_1, \ldots, x_n), HLL(y_1, \ldots, y_n)\) it is easy to merge them to give \(HLL(x_1, \ldots, x_n, y_1, \ldots, y_n)\).

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- Set the maximum \( \# \) of trailing zeros to the maximum in the two sketches.

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Questions on distinct elements counting?