Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov’s inequality)
- 2-universal and pairwise independent hash functions
- Chebyshev: \( \Pr(|X - \mathbb{E}[X]| \geq t) \leq \text{Var}[X]/t^2 \)
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This Time:

- Random hashing for load balancing. Motivating:
  - Stronger concentration inequalities: Chebyshev’s inequality, exponential tail bounds, and their connections to the law of large numbers and central limit theorem.
  - The union bound.
Randomized Load Balancing:

- $n$ requests randomly assigned to $k$ servers.

\[ \text{Expected load and variance for server } i \text{ is } E[R_i] = \frac{n}{k} \text{ and } \text{Var}[R_i] = \frac{n}{k^2} \left(1 - \frac{1}{k}\right). \]

- Suppose each server can handle at most $E[R_i] = \frac{n}{k}$ requests.

- By Markov’s inequality, $\Pr[R_i \geq 2 \cdot E[R_i]] \leq \frac{1}{2}$.

- By Chebyshev’s inequality, $\Pr[R_i \geq 2 \cdot E[R_i]] \leq \frac{\text{Var}[R_i]}{E[R_i]^2} < \frac{k}{n}$. 

Diagram:
- Client Requests
- Routers
- Server 1
- Server 2
- ... Server k
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- By Chebyshev’s inequality, $\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq \frac{\text{Var}[R_i]}{\mathbb{E}[R_i]^2} < \frac{k}{n}$. 
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$n$: total number of requests, $k$: number of servers randomly assigned requests, $R_i$: number of requests assigned to server $i$. $\mathbb{E}[R_i] = \frac{n}{k}$. $\text{Var}[R_i] = n/k$. 
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$$\Pr \left( \max_i (R_i) \geq \frac{2n}{k} \right) = \Pr \left( \left[ R_1 \geq \frac{2n}{k} \right] \cup \left[ R_2 \geq \frac{2n}{k} \right] \cup \ldots \cup \left[ R_k \geq \frac{2n}{k} \right] \right)$$

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We want to show that $\Pr \left( \bigcup_{i=1}^{k} \left[ R_i \geq \frac{2n}{k} \right] \right)$ is small.

How do we do this? Note that $R_1, \ldots, R_k$ are correlated in a somewhat complex way.

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Union Bound: For any random events $A_1, A_2, ..., A_k$, 

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\Pr(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \Pr(A_1) + \Pr(A_2) + \ldots + \Pr(A_k).
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On the first problem set, you will prove the union bound, as a consequence of Markov’s inequality.
What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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Applying the Union Bound

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As long as $k \ll \sqrt{n}$, the maximum server load will be small (compared to the expected load) with good probability.

$n$: total number of requests, \( k \): number of servers randomly assigned requests, \( R_i \): number of requests assigned to server \( i \). \( \mathbb{E}[R_i] = \frac{n}{k} \). \( \text{Var}[R_i] = \frac{n}{k} \).
BACK TO CHEBYSHEV’S INEQUALITY

\[ \Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2} \]

\(X\): any random variable, \(t, s\): any fixed numbers.
BACK TO CHEBYSHEV’S INEQUALITY

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What is the probability that \( X \) falls \( s \) standard deviations from it’s mean?

\[ \text{Var}[X] = \frac{1}{s^2} \]

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Pr(\(|X - \mathbb{E}[X]| \geq s \cdot \sqrt{\text{Var}[X]}\)) \leq \frac{\text{Var}[X]}{s^2 \cdot \text{Var}[X]} = \frac{1}{s^2}.$
Back to Chebyshev’s Inequality

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Why is this so powerful?

\( X \): any random variable, \( t, s \): any fixed numbers.
Consider drawing independent identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with mean $\mu$ and variance $\sigma^2$.
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By Chebyshev’s Inequality: for any fixed value $\epsilon > 0$,

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- Cannot show from vanilla Markov’s inequality.
The number of servers must be small compared to the number of requests ($k = O(\sqrt{n})$) for the maximum load to be bounded in comparison to the expected load with good probability.

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- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.

\( n \): total number of requests, \( k \): number of servers randomly assigned requests.
Questions on union bound, Chebyshev’s inequality, random hashing?
We flip $n = 100$ independent coins, each are heads with probability $1/2$ and tails with probability $1/2$. Let $H$ be the number of heads.
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\mathbb{E}[H] = \frac{n}{2} = 50 \quad \text{and} \quad \text{Var}[H] = \frac{n}{4} = 25
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**Markov’s:**

- $\Pr(H \geq 60) \leq .833$
- $\Pr(H \geq 70) \leq .714$
- $\Pr(H \geq 80) \leq .625$
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### Markov’s:
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### Chebyshev’s:
- $\Pr(H \geq 60) \leq .25$
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- $\Pr(H \geq 80) \leq .0278$
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<table>
<thead>
<tr>
<th>Markov’s:</th>
<th>Chebyshev’s:</th>
<th>In Reality:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pr(H \geq 60) \leq .833 )</td>
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<td>( \Pr(H \geq 60) = 0.0284 )</td>
</tr>
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<td>( \Pr(H \geq 80) &lt; 10^{-9} )</td>
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\( H \) has a simple Binomial distribution, so can compute these probabilities exactly.
To be fair.... Markov and Chebyshev’s inequalities apply much more generally than to Binomial random variables like coin flips.
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- Markov’s: \( \Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t} \). First Moment.
- Chebyshev’s: \( \Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2} \). Second Moment.
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Can we obtain tighter concentration bounds that still apply to very general distributions?

• Markov’s: \( \Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t} \). First Moment.

• Chebyshev’s: \( \Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2} \). Second Moment.

• What if we just apply Markov’s inequality to even higher moments?
Consider any random variable $X$:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$
A FOURTH MOMENT BOUND

Consider any random variable $X$:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr \left( (X - \mathbb{E}[X])^4 \geq t^4 \right) \leq \frac{\mathbb{E} \left[ (X - \mathbb{E}[X])^4 \right]}{t^4}.$$
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**Application to Coin Flips:** Recall: $n = 100$ independent fair coins, $H$ is the number of heads.

- Bound the fourth moment:
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$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right]$$

where $H_i = 1$ if coin flip $i$ is heads and 0 otherwise.
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- Apply Fourth Moment Bound: $\Pr(|H - \mathbb{E}[H]| \geq t) \leq \frac{1862.5}{t^4}$. 
TIGHTER BOUNDS

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\[ 4^{th} \text{ Moment:} \]  
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Can we just keep applying Markov’s inequality to higher and higher moments and getting tighter bounds?  
• Yes! To a point.  
• In fact – don’t need to just apply Markov’s to \( |X - E[X]| \) for some \( k \). Can apply to any monotonic function \( f(|X - E[X]|) \).  
• Why monotonic?  
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$$M_t(X) = e^{t \cdot (X - \mathbb{E}[X])}$$
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**EXPONENTIAL CONCENTRATION BOUNDS**

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- We will explore the basic proof approach in homework.
**Bernstein Inequality:** Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \text{Var}[X_i]$. For any $t \geq 0$:

$$
\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq t \right) \leq 2 \exp \left( -\frac{t^2}{2\sigma^2 + \frac{4}{3} Mt} \right).
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- An exponentially stronger dependence on \(s\)!
Consider again bounding the number of heads $H$ in $n = 100$ independent coin flips.

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$H$: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$. 
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Getting much closer to the true probability.

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A useful variation of the Bernstein inequality for binary (indicator) random variables is:

**Chernoff Bound (simplified version):** Consider independent random variables \( X_1, \ldots, X_n \) taking values in \( \{0, 1\} \). Let \( \mu = \mathbb{E}[\sum_{i=1}^n X_i] \). For any \( \delta \geq 0 \)

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As $\delta$ gets larger and larger, the bound falls off exponentially fast.