COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor
Lecture 4
Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov’s inequality)
- 2-universal and pairwise independent hash functions
- Chebyshev: $\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2}$
LAST TIME

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This Time:

- Random hashing for load balancing. Motivating:
  - Stronger concentration inequalities: Chebyshev’s inequality, exponential tail bounds, and their connections to the law of large numbers and central limit theorem.
  - The union bound.
Randomized Load Balancing:

- $n$ requests randomly assigned to $k$ servers.

Expected load and variance for server $i$:

- $E[R_i] = \frac{n}{k}$ and $\text{Var}[R_i] = \frac{n}{k} \left(1 - \frac{1}{k}\right)$. 

By Markov's inequality:

- $\Pr[R_i \geq 2E[R_i]] \leq \frac{1}{2}$.

By Chebyshev's inequality:

- $\Pr[R_i \geq 2E[R_i]] \leq \frac{\text{Var}[R_i]}{E[R_i]^2} < \frac{k}{n}$. 

Diagram:

- Client Requests
- Routers
- Server 1, Server 2, ..., Server k

Diagram shows the flow of client requests to multiple servers through routers.
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- Suppose each server can handle at most $\mathbb{E}[R_i] = n/k$ requests
- By Markov’s inequality, $\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq 1/2$.
- By Chebyshev’s inequality, $\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq \frac{\text{Var}[R_i]}{(\mathbb{E}[R_i])^2} < \frac{k}{n}$. 
What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$n$: total number of requests, $k$: number of servers randomly assigned requests, $R_i$: number of requests assigned to server $i$. $\mathbb{E}[R_i] = \frac{n}{k}$. Var$[R_i] = n/k$. 
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\Pr\left(\max_i (R_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[ R_1 \geq \frac{2n}{k}\right] \cup \left[ R_2 \geq \frac{2n}{k}\right] \cup \ldots \cup \left[ R_k \geq \frac{2n}{k}\right]\right)
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We want to show that $\Pr\left(\bigcup_{i=1}^{k} [R_i \geq \frac{2n}{k}]\right)$ is small.

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How do we do this? Note that \(R_1, \ldots, R_k\) are correlated in a somewhat complex way.

---

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**Union Bound**: For any random events $A_1, A_2, ..., A_k$,

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \Pr(A_1) + \Pr(A_2) + \ldots + \Pr(A_k).$$

When is the union bound tight? When $A_1, \ldots, A_k$ are all disjoint.

On the first problem set, you will prove the union bound, as a consequence of Markov's inequality.
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As long as \(k \ll \sqrt{n}\), the maximum server load will be small (compared to the expected load) with good probability.

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back to chebyshev’s inequality

\[ \Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2} \]

\(X\): any random variable, \(t, s\): any fixed numbers.
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Why is this so powerful?

$X$: any random variable, $t, s$: any fixed numbers.
Law of Large Numbers

Consider drawing independent identically distributed (i.i.d.)
random variables $X_1, \ldots, X_n$ with mean $\mu$ and variance $\sigma^2$. 

Law of Large Numbers: with enough samples $n$, the sample
average will always concentrate to the mean.

• Cannot show from vanilla Markov’s inequality.
Consider drawing independent identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with mean $\mu$ and variance $\sigma^2$.

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The number of servers must be small compared to the number of requests \( (k = O(\sqrt{n})) \) for the maximum load to be bounded in comparison to the expected load with good probability.

- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.

\[ n: \text{total number of requests}, \quad k: \text{number of servers randomly assigned requests}. \]
Questions on union bound, Chebyshev’s inequality, random hashing?
We flip $n = 100$ independent coins, each are heads with probability $1/2$ and tails with probability $1/2$. Let $H$ be the number of heads.
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$$\mathbb{E}[H] = \frac{n}{2} = 50$$ and

$$\text{Var}[H] =$$
We flip \( n = 100 \) independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let \( H \) be the number of heads.

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\mathbb{E}[H] = \frac{n}{2} = 50 \quad \text{and} \quad \text{Var}[H] = \frac{n}{4} = 25
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**Markov’s:**

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\begin{align*}
\Pr(H \geq 60) &\leq .833 \\
\Pr(H \geq 70) &\leq .714 \\
\Pr(H \geq 80) &< 10^{-9}
\end{align*}
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FLIPPING COINS

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### Markov’s:
- \( \Pr(H \geq 60) \leq .833 \)
- \( \Pr(H \geq 70) \leq .714 \)
- \( \Pr(H \geq 80) \leq .625 \)

### Chebyshev’s:
- \( \Pr(H \geq 60) \leq .25 \)
- \( \Pr(H \geq 70) \leq .0625 \)
- \( \Pr(H \geq 80) \leq .0278 \)
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$E[H] = \frac{n}{2} = 50$ and $\text{Var}[H] = \frac{n}{4} = 25 \rightarrow s.d. = 5$

<table>
<thead>
<tr>
<th>Markov’s:</th>
<th>Chebyshev’s:</th>
<th>In Reality:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pr(H \geq 60) \leq .833$</td>
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$H$ has a simple Binomial distribution, so can compute these probabilities exactly.
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• What if we just apply Markov’s inequality to even higher moments?
Consider any random variable $X$:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$
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A FOURTH MOMENT BOUND

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**Application to Coin Flips:** Recall: $n = 100$ independent fair coins, $H$ is the number of heads.

- Bound the fourth moment:
Consider any random variable $X$:

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- Apply Fourth Moment Bound: $\Pr \left( |H - \mathbb{E}[H]| \geq t \right) \leq \frac{1862.5}{t^4}$. 


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- In fact – don’t need to just apply Markov’s to \( |X - \mathbb{E}[X]|^k \) for some \( k \). Can apply to any monotonic function \( f (|X - \mathbb{E}[X]|) \).
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- Why monotonic?
  \[
  \Pr(|X - \mathbb{E}[X]| > t) = \Pr(f(|X - \mathbb{E}[X]|) > f(t)).
  \]
Moment Generating Function: Consider for any $t > 0$:

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- We will explore the basic proof approach in homework.
Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \text{Var}[X_i]$. For any $t \geq 0$:

$$
\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq t \right) \leq 2 \exp \left( - \frac{t^2}{2\sigma^2 + \frac{4}{3}Mt} \right).
$$

Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$. Compare to Chebyshev’s:

$$
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$$
\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \geq s \sigma \right) \leq 2 \exp \left( -\frac{s^2}{4} \right).
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Consider again bounding the number of heads $H$ in $n = 100$ independent coin flips.

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Getting much closer to the true probability.

$H$: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$. 
A useful variation of the Bernstein inequality for binary (indicator) random variables is:

**Chernoff Bound (simplified version):** Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$. For any $\delta \geq 0$

$$\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \geq \delta \mu \right) \leq 2 \exp \left( -\frac{\delta^2 \mu}{2 + \delta} \right).$$
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As $\delta$ gets larger and larger, the bound falls off exponentially fast.
We hash \( m \) values \( x_1, \ldots, x_m \) using a random hash function into a table with \( n = m \) entries.
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What will be the maximum number of items hashed into the same location?
Let $S_i$ be the number of items hashed into position $i$ and $S_{i,j}$ be 1 if $x_j$ is hashed into bucket $i$ ($h(x_j) = i$) and 0 otherwise.

$m$: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items.  
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Let $S_i$ be the number of items hashed into position $i$ and $S_{i,j}$ be 1 if $x_j$ is hashed into bucket $i$ ($h(x_j) = i$) and 0 otherwise.

$$
\mathbb{E}[S_i] = \sum_{j=1}^{m} \mathbb{E}[S_{i,j}] = m \cdot \frac{1}{m} = 1
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$$

**By the Chernoff Bound:** for any $\delta \geq 0$,

$$
Pr(S_i \geq 1 + \delta) \leq Pr \left( \left| \sum_{i=1}^{n} S_{i,j} - 1 \right| \geq \delta \right) \leq 2 \exp \left( -\frac{\delta^2}{2 + \delta} \right)
$$

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Pr(S_i ≥ 1 + δ) ≤ Pr \left( \left| \sum_{i=1}^{n} S_{i,j} - 1 \right| ≥ δ \right) ≤ 2 \exp \left( -\frac{\delta^2}{2 + \delta} \right).
\[ \Pr(S_i \geq 1 + \delta) \leq \Pr \left( \left| \sum_{i=1}^{n} S_{i,j} - 1 \right| \geq \delta \right) \leq 2 \exp \left( -\frac{\delta^2}{2 + \delta} \right). \]

Set \( \delta = 20 \log m \). Gives:

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Set \(\delta = 20 \log m\). Gives:

Pr\((S_i \geq 20 \log m + 1) \leq 2 \exp\left(-\frac{(20 \log m)^2}{2 + 20 \log m}\right)\).
MAXIMUM LOAD IN RANDOMIZED HASHING

\[
\Pr(S_i \geq 1 + \delta) \leq \Pr \left( \left| \sum_{i=1}^{n} S_{i,j} - 1 \right| \geq \delta \right) \leq 2 \exp \left( -\frac{\delta^2}{2 + \delta} \right).
\]

Set \( \delta = 20 \log m \). Gives:

\[
\Pr(S_i \geq 20 \log m + 1) \leq 2 \exp \left( -\frac{(20 \log m)^2}{2 + 20 \log m} \right) \leq \exp(-18 \log m) \leq \frac{2}{m^{18}}.
\]

**Apply Union Bound:**

\[
\Pr(\max_{i \in [m]} S_i \geq 20 \log m + 1) = \Pr \left( \bigcup_{i=1}^{m} (S_i \geq 20 \log m + 1) \right).
\]

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\[ \Pr(S_i \geq 1 + \delta) \leq \Pr \left( \left| \sum_{i=1}^{n} S_{i,j} - 1 \right| \geq \delta \right) \leq 2 \exp \left( -\frac{\delta^2}{2 + \delta} \right). \]

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\[ \Pr(S_i \geq 20 \log m + 1) \leq 2 \exp \left( - \frac{(20 \log m)^2}{2 + 20 \log m} \right) \leq \exp(-18 \log m) \leq \frac{2}{m^{18}}. \]

**Apply Union Bound:**

\[ \Pr(\max_{i \in [m]} S_i \geq 20 \log m + 1) = \Pr \left( \bigcup_{i=1}^{m} (S_i \geq 20 \log m + 1) \right) \leq \sum_{i=1}^{m} \Pr(S_i \geq 20 \log m + 1). \]

\( m \): total number of items hashed and size of hash table. \( S_i \): number of items hashed to bucket \( i \). \( S_{i,j} \): indicator if \( x_j \) is hashed to bucket \( i \). \( \delta \): any value \( \geq 0 \).
Pr($S_i \geq 1 + \delta$) $\leq$ Pr\left(\left|\sum_{i=1}^{n} S_{i,j} - 1\right| \geq \delta\right) $\leq$ 2 exp \left(-\frac{\delta^2}{2 + \delta}\right).

Set $\delta = 20 \log m$. Gives:

Pr($S_i \geq 20 \log m + 1$) $\leq$ 2 exp \left(-\frac{(20 \log m)^2}{2 + 20 \log m}\right) $\leq$ exp(-18 log m) $\leq$ $\frac{2}{m^{18}}$.

**Apply Union Bound:**

Pr($\max_{i \in [m]} S_i \geq 20 \log m + 1$) = Pr\left(\bigcup_{i=1}^{m} (S_i \geq 20 \log m + 1)\right)

\leq \sum_{i=1}^{m} \Pr(S_i \geq 20 \log m + 1) \leq m \cdot \frac{2}{m^{18}} = \frac{2}{m^{17}}.

$m$: total number of items hashed and size of hash table. $S_i$: number of items hashed to bucket $i$. $S_{i,j}$: indicator if $x_j$ is hashed to bucket $i$. $\delta$: any value $\geq 0$. 


Upshot: If we randomly hash \( m \) items into a hash table with \( m \) entries the maximum load per bucket is \( O(\log m) \) with very high probability.
Upshot: If we randomly hash $m$ items into a hash table with $m$ entries the maximum load per bucket is $O(\log m)$ with very high probability.

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**Upshot:** If we randomly hash $m$ items into a hash table with $m$ entries the maximum load per bucket is $O(\log m)$ with very high probability.

- So, even with a simple linked list to store the items in each bucket, worst case query time is $O(\log m)$.
- Using Chebyshev’s inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability.
Upshot: If we randomly hash $m$ items into a hash table with $m$ entries the maximum load per bucket is $O(\log m)$ with very high probability.

- So, even with a simple linked list to store the items in each bucket, worst case query time is $O(\log m)$.
- Using Chebyshev’s inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability.
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a $k$-wise independent hash function for $k = O(\log m)$.