COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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Lecture 11
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- A 3 minute Youtube clip with a resolution of $500 \times 500$ pixels at 15 frames/second with 3 color channels is a recording of \( \geq 2 \text{ billion pixel values} \). Even a $500 \times 500$ pixel color image has 750,000 pixel values.
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- The human genome contains 3 billion+ base pairs. Genetic datasets often contain information on 100s of thousands+ mutations and genetic markers.
In data analysis and machine learning, data points with many attributes are often stored, processed, and interpreted as **high dimensional vectors**, with real valued entries.
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**ATAGCGGTAGT**  \[\mathbf{x} = [1 \ 2 \ 1 \ 3 \ 4 \ 4 \ 3 \ 2 \ 1 \ 3 \ 4]\]

**x**  \[x = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1...]\]
In data analysis and machine learning, data points with many attributes are often stored, processed, and interpreted as high dimensional vectors, with real valued entries.

Similarities/distances between vectors (e.g., $\langle x, y \rangle$, $\|x - y\|_2$) have meaning for underlying data points.
Data points are interpreted as high dimensional vectors, with real valued entries. Data set is interpreted as a matrix.

**Data Points:** \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathbb{R}^d \).

**Data Set:** \( X \in \mathbb{R}^{n \times d} \) with \( i^{th} \) rows equal to \( \vec{x}_i \).
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Many data points \( n \Rightarrow \) tall. Many dimensions \( d \Rightarrow \) wide.
Dimensionality Reduction: Compress data points so that they lie in many fewer dimensions.
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\[
\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d \rightarrow \tilde{\vec{x}}_1, \ldots, \tilde{\vec{x}}_n \in \mathbb{R}^m \text{ for } m \ll d.
\]

\[\begin{array}{l}
x = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ldots] \quad \rightarrow \quad \tilde{x} = [-5.5 \ 4 \ 3.2 \ -1]
\end{array}\]
**Dimensionality Reduction**: Compress data points so that they lie in many fewer dimensions.

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`Lossy compression' that still preserves important information about the relationships between \( \tilde{\vec{x}}_1, \ldots, \tilde{\vec{x}}_n \).

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Generally will not consider directly how well \( \tilde{\vec{x}}_i \) approximates \( \vec{x}_i \).
**Low Distortion Embedding:** Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$, distance function $D$, and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) and distance function $\tilde{D}$ such that for all $i, j \in [n]$:

$$(1 - \epsilon)D(\vec{x}_i, \vec{x}_j) \leq \tilde{D}(\tilde{x}_i, \tilde{x}_j) \leq (1 + \epsilon)D(\vec{x}_i, \vec{x}_j).$$
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Have already seen one example in class: **MinHash**.

\[\begin{align*}
\text{x}_A &= [1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0] \\
\text{x}_B &= [0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1]
\end{align*}\]
**Low Distortion Embedding:** Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$, distance function $D$, and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) and distance function $\tilde{D}$ such that for all $i, j \in [n]$:

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\begin{align*}
x_A &= [1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0] \\
x_B &= [0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1] \\
\tilde{x}_A &= [.12] \\
\tilde{x}_B &= [.18]
\end{align*}
\]
Low Distortion Embedding: Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$, distance function $D$, and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) and distance function $\tilde{D}$ such that for all $i, j \in [n]$:

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Low Distortion Embedding: Given \( \vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d \), distance function \( D \), and error parameter \( \epsilon \geq 0 \), find \( \tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m \) (where \( m \ll d \)) and distance function \( \tilde{D} \) such that for all \( i, j \in [n] \):

\[
(1 - \epsilon)D(\vec{x}_i, \vec{x}_j) \leq \tilde{D}(\tilde{x}_i, \tilde{x}_j) \leq (1 + \epsilon)D(\vec{x}_i, \vec{x}_j).
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With large enough signature size \( r \), \[
\frac{\# \text{ matching entries in } \tilde{x}_A, \tilde{x}_B}{r} \approx J(\tilde{x}_A, \tilde{x}_B).
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**Low Distortion Embedding:*** Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$, distance function $D$, and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) and distance function $\tilde{D}$ such that for all $i, j \in [n]$:

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Have already seen one example in class: **MinHash.**

![Diagram](image)

With large enough signature size $r$, $\frac{\# \text{ matching entries in } \tilde{x}_A, \tilde{x}_B}{r} \approx J(\tilde{x}_A, \tilde{x}_B)$.

- Note: here $J(\tilde{x}_A, \tilde{x}_B)$ is a similarity rather than a distance. So this is not quite low distortion embedding, but is closely related.
Euclidean Low Distortion Embedding: Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) such that for all $i, j \in [n]$:

$$(1 - \epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{x}_i - \tilde{x}_j\|_2 \leq (1 + \epsilon)\|\vec{x}_i - \vec{x}_j\|_2.$$ 

Recall that for $\vec{z} \in \mathbb{R}^n$, $\|\vec{z}\|_2 = \sqrt{\sum_{i=1}^{n} z(i)^2}$. 


**Euclidean Low Distortion Embedding**: Given $\bar{x}_1, \ldots, \bar{x}_n \in \mathbb{R}^d$ and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) such that for all $i, j \in [n]$:

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Recall that for $\bar{z} \in \mathbb{R}^n$, $\| \bar{z} \|_2 = \sqrt{\sum_{i=1}^{n} \bar{z}(i)^2}$. 

**Pythagorean theorem.**

$\| \bar{z} \|_2 = \sqrt{z(1)^2 + z(2)^2}$
Euclidean Low Distortion Embedding: Given \( \vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d \) and error parameter \( \epsilon \geq 0 \), find \( \tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m \) (where \( m \ll d \)) such that for all \( i, j \in [n] \):

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A very easy case: Assume that $\vec{x}_1, \ldots, \vec{x}_n$ all lie on the 1\textsuperscript{st} axis in $\mathbb{R}^d$. 

![Diagram of points on the 1\textsuperscript{st} axis in $\mathbb{R}^d$.]
A very easy case: Assume that $\vec{x}_1, \ldots, \vec{x}_n$ all lie on the $1^{st}$ axis in $\mathbb{R}^d$.

Set $m = 1$ and $\tilde{x}_i = [\vec{x}_i(1)]$ (i.e., $\tilde{x}_i$ contains just a single number).

$\|\tilde{x}_i - \tilde{x}_j\|_2 = \sqrt{[\vec{x}_i(1) - \vec{x}_j(1)]^2} = |\vec{x}_i(1) - \vec{x}_j(1)| = \|\vec{x}_i - \vec{x}_j\|_2$. 
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- An embedding with no distortion from any $d$ into $m = 1$. 
EMBEDDING WITH ASSUMPTIONS

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- $\|\tilde{x}_i - \tilde{x}_j\|_2 = \sqrt{[\vec{x}_i(1) - \vec{x}_j(1)]^2} = |\vec{x}_i(1) - \vec{x}_j(1)| = \|\vec{x}_i - \vec{x}_j\|_2$.
- An embedding with no distortion from any $d$ into $m = 1$.
- More generally, there’s a no distortion embedding into $m = D$ dimensions if all the points lie is a $D$ dimensional space.
What about when we don’t make any assumptions on $\vec{x}_1, \ldots, \vec{x}_n$.
I.e., they can be scattered arbitrarily around $d$-dimensional space?

- Can we find a no-distortion embedding into $m \ll d$ dimensions?
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- Can we find a no-distortion embedding into $m \ll d$ dimensions? 
  No. Require $m = d$.
- Can we find an $\epsilon$-distortion embedding into $m \ll d$ dimensions for $\epsilon > 0$?

  For all $i, j$: $(1 - \epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{x}_i - \tilde{x}_j\|_2 \leq (1 + \epsilon)\|\vec{x}_i - \vec{x}_j\|_2$. 

EMBEDDING WITH NO ASSUMPTIONS
What about when we don’t make any assumptions on \( \vec{x}_1, \ldots, \vec{x}_n \). I.e., they can be scattered arbitrarily around \( d \)-dimensional space?

- Can we find a no-distortion embedding into \( m \ll d \) dimensions? No. Require \( m = d \).
- Can we find an \( \epsilon \)-distortion embedding into \( m \ll d \) dimensions for \( \epsilon > 0 \)? Yes! Always, with \( m \) depending on \( \epsilon \).

For all \( i, j \): \( (1 - \epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{x}_i - \tilde{x}_j\|_2 \leq (1 + \epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \).
**Johnson-Lindenstrauss Lemma:** For any set of points $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and $\epsilon > 0$ there exists a linear map $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^m$ such that $m = O \left( \frac{\log n}{\epsilon^2} \right)$ and letting $\tilde{x}_i = \Pi \vec{x}_i$:

For all $i, j$:

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Further, if $\Pi \in \mathbb{R}^{m \times d}$ has each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, it satisfies the guarantee with high probability.
Johnson-Lindenstrauss Lemma: For any set of points $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and $\epsilon > 0$ there exists a linear map $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^m$ such that $m = O\left(\frac{\log n}{\epsilon^2}\right)$ and letting $\tilde{x}_i = \Pi \vec{x}_i$:

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Further, if $\Pi \in \mathbb{R}^{m \times d}$ has each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, it satisfies the guarantee with high probability.

For $d = 1$ trillion, $\epsilon = .05$, and $n = 100,000$, $m \approx 6600$. 
THE JOHNSON-LINDENSTRAUSS Lemma: For any set of points \( \vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d \) and \( \epsilon > 0 \) there exists a linear map \( \Pi : \mathbb{R}^d \rightarrow \mathbb{R}^m \) such that \( m = O\left(\frac{\log n}{\epsilon^2}\right) \) and letting \( \tilde{x}_i = \Pi \vec{x}_i \):

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For \( d = 1 \) trillion, \( \epsilon = .05 \), and \( n = 100,000 \), \( m \approx 6600 \).

Very surprising! Powerful result with a simple construction: applying a random linear transformation to a set of points preserves distances between all those points with high probability.
For any $\vec{x}_1, \ldots, \vec{x}_n$ and $\Pi \in \mathbb{R}^{m \times d}$ with each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, with high probability, letting $x_i = \Pi \vec{x}_i$:

For all $i, j$:

$$ (1 - \epsilon) \| \vec{x}_i - \vec{x}_j \|_2 \leq \| x_i - x_j \|_2 \leq (1 + \epsilon) \| \vec{x}_i - \vec{x}_j \|_2. $$

$\Pi$ is known as a random projection. It is a random linear function, mapping length $d$ vectors to length $m$ vectors.

$\Pi$ is data oblivious. Stark contrast to methods like PCA.
For any $\vec{x}_1, \ldots, \vec{x}_n$ and $\Pi \in \mathbb{R}^{m \times d}$ with each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, with high probability, letting $x_i = \Pi \vec{x}_i$:

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Data oblivious property means that once \( \Pi \) is chosen, \( x_1, \ldots, x_n \) can be computed in a stream with little memory.

Storage is just \( O(nm) \) rather than \( O(nd) \).
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• Many alternative constructions: $\pm 1$ entries, sparse (most entries 0), Fourier structured, etc. $\implies$ more efficient computation of $x_i = \prod \tilde{x}_i$.

• Data oblivious property means that once $\prod$ is chosen, $x_1, \ldots, x_n$ can be computed in a stream with little memory.

• Storage is just $O(nm)$ rather than $O(nd)$.

• Compression can be performed in parallel on different servers.

• When new data points are added, can be easily compressed, without updating existing points.
The Johnson-Lindenstrauss Lemma is a direct consequence of a closely related lemma:

**Distributional JL Lemma:** Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\mathbf{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1 - \epsilon)\|\mathbf{y}\|_2 \leq \|\mathbf{\Pi}\mathbf{y}\|_2 \leq (1 + \epsilon)\|\mathbf{y}\|_2$$

**\(\mathbf{\Pi} \in \mathbb{R}^{m \times d}\):** random projection matrix. **\(d\):** original dimension. **\(m\):** compressed dimension, **\(\epsilon\):** embedding error, **\(\delta\):** embedding failure prob.
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Applying a random matrix $\mathbf{\Pi}$ to any vector $\mathbf{y}$ preserves $\mathbf{y}$’s norm with high probability.

$\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. $d$: original dimension. $m$: compressed dimension, $\epsilon$: embedding error, $\delta$: embedding failure prob.
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(1 - \epsilon) \|\mathbf{y}\|_2 \leq \|\mathbf{\Pi}\mathbf{y}\|_2 \leq (1 + \epsilon) \|\mathbf{y}\|_2
\]

Applying a random matrix \( \mathbf{\Pi} \) to any vector \( \mathbf{y} \) preserves \( \mathbf{y} \)'s norm with high probability.

- Like a low-distortion embedding, but for the length of a compressed vector rather than distances between vectors.

**\( \mathbf{\Pi} \in \mathbb{R}^{m \times d} \):** random projection matrix. **\( d \):** original dimension. **\( m \):** compressed dimension, **\( \epsilon \):** embedding error, **\( \delta \):** embedding failure prob.
Distributional JL Lemma $\Rightarrow$ JL Lemma: Distributional JL show that a random projection $\Pi$ preserves the norm of any $y$. The main JL Lemma says that $\Pi$ preserves distances between vectors.

$\vec{x}_1, \ldots, \vec{x}_n$: original points, $x_1, \ldots, x_n$: compressed points, $\Pi \in \mathbb{R}^{m \times d}$: random projection matrix. $d$: original dimension. $m$: compressed dimension, $\epsilon$: embedding error, $\delta$: embedding failure prob.
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**Proof:** Given $\vec{x}_1, \ldots, \vec{x}_n$, define $\binom{n}{2}$ vectors $\vec{y}_{ij}$ where $\vec{y}_{ij} = \vec{x}_i - \vec{x}_j$. 

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Proof: Given $\vec{x}_1, \ldots, \vec{x}_n$, define $\binom{n}{2}$ vectors $\vec{y}_{ij}$ where $\vec{y}_{ij} = \vec{x}_i - \vec{x}_j$.  

\[ \vec{x}_1, \ldots, \vec{x}_n: \text{original points}, \quad \vec{x}_1, \ldots, \vec{x}_n: \text{compressed points}, \quad \Pi \in \mathbb{R}^{m \times d}: \text{random projection matrix}. \quad d: \text{original dimension}. \quad m: \text{compressed dimension}, \epsilon: \text{embedding error}, \delta: \text{embedding failure prob.} \]
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- If we choose $\Pi$ with $m = O\left(\frac{\log 1/\delta'}{\epsilon^2}\right)$, for each $\vec{y}_{ij}$ with probability $\geq 1 - \delta'$ we have:

$$
(1 - \epsilon)\|\vec{y}_{ij}\|_2 \leq \|\Pi\vec{y}_{ij}\|_2 \leq (1 + \epsilon)\|\vec{y}_{ij}\|_2
$$

$x_1, \ldots, x_n$: original points, $\tilde{x}_1, \ldots, \tilde{x}_n$: compressed points, $\Pi \in \mathbb{R}^{m \times d}$: random projection matrix. $d$: original dimension. $m$: compressed dimension, $\epsilon$: embedding error, $\delta$: embedding failure prob.
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$$
(1 - \epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\mathbf{\Pi}(\vec{x}_i - \vec{x}_j)\|_2 \leq (1 + \epsilon)\|\vec{x}_i - \vec{x}_j\|_2
$$

$\vec{x}_1, \ldots, \vec{x}_n$: original points, $x_1, \ldots, x_n$: compressed points, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. $d$: original dimension. $m$: compressed dimension, $\epsilon$: embedding error, $\delta$: embedding failure prob.
**Distributional JL Lemma** \(\implies\) **JL Lemma:** Distributional JL show that a random projection \(\Pi\) preserves the norm of any \(y\). The main JL Lemma says that \(\Pi\) preserves distances between vectors.

Since \(\Pi\) is linear these are the same thing!

**Proof:** Given \(\tilde{x}_1, \ldots, \tilde{x}_n\), define \(\binom{n}{2}\) vectors \(\tilde{y}_{ij}\) where \(\tilde{y}_{ij} = \tilde{x}_i - \tilde{x}_j\).

- If we choose \(\Pi\) with \(m = O\left(\frac{\log 1/\delta'}{\epsilon^2}\right)\), for each \(\tilde{y}_{ij}\) with probability \(\geq 1 - \delta'\) we have:

\[
(1 - \epsilon)\|\tilde{x}_i - \tilde{x}_j\|_2 \leq \|x_i - x_j\|_2 \leq (1 + \epsilon)\|\tilde{x}_i - \tilde{x}_j\|_2
\]

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\(\tilde{x}_1, \ldots, \tilde{x}_n\): original points, \(x_1, \ldots, x_n\): compressed points, \(\Pi \in \mathbb{R}^{m \times d}\): random projection matrix. \(d\): original dimension. \(m\): compressed dimension, \(\epsilon\): embedding error, \(\delta\): embedding failure prob.
Claim: If we choose $\Pi$ with i.i.d. $\mathcal{N}(0, 1/m)$ entries and $m = O \left( \frac{\log(1/\delta')}{\epsilon^2} \right)$, letting $x_i = \Pi \bar{x}_i$, for each pair $\bar{x}_i, \bar{x}_j$ with probability $\geq 1 - \delta'$ we have:

$$(1 - \epsilon) \| \bar{x}_i - \bar{x}_j \|_2 \leq \| x_i - x_j \|_2 \leq (1 + \epsilon) \| \bar{x}_i - \bar{x}_j \|_2.$$
**Claim:** If we choose $\Pi$ with i.i.d. $\mathcal{N}(0, 1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, letting $x_i = \Pi \tilde{x}_i$, for each pair $\tilde{x}_i, \tilde{x}_j$ with probability $\geq 1 - \delta'$ we have:

$$(1 - \epsilon)\|\tilde{x}_i - \tilde{x}_j\|_2 \leq \|x_i - x_j\|_2 \leq (1 + \epsilon)\|\tilde{x}_i - \tilde{x}_j\|_2.$$ 

With what probability are all pairwise distances preserved?

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With what probability are all pairwise distances preserved?

Union bound: With probability $\geq 1 - \left( \binom{n}{2} \right) \cdot \delta'$ all pairwise distances are preserved.

$\bar{x}_1, \ldots, \bar{x}_n$: original points, $x_1, \ldots, x_n$: compressed points, $\Pi \in \mathbb{R}^{m \times d}$: random projection matrix. $d$: original dimension. $m$: compressed dimension, $\epsilon$: embedding error, $\delta$: embedding failure prob.
**Claim:** If we choose $\Pi$ with i.i.d. $N(0, 1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, letting $x_i = \Pi \vec{x}_i$, for each pair $\vec{x}_i, \vec{x}_j$ with probability $\geq 1 - \delta'$ we have:

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With what probability are all pairwise distances preserved?

**Union bound:** With probability $\geq 1 - \binom{n}{2} \cdot \delta'$ all pairwise distances are preserved.

Apply the claim with $\delta' = \delta/\binom{n}{2}$. 

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**DISTRIBUTIONAL JL \implies JL**

**Claim:** If we choose $\Pi$ with i.i.d. $\mathcal{N}(0, 1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, letting $x_i = \Pi \vec{x}_i$, for each pair $\vec{x}_i, \vec{x}_j$ with probability $\ge 1 - \delta'$ we have:

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\[\vec{x}_1, \ldots, \vec{x}_n: \text{original points}, \; x_1, \ldots, x_n: \text{compressed points}, \; \Pi \in \mathbb{R}^{m \times d}: \text{random projection matrix.} \; \epsilon: \text{embedding error, } \delta: \text{embedding failure prob.} \]
**Claim:** If we choose $\Pi$ with i.i.d. $\mathcal{N}(0,1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, letting $x_i = \Pi \tilde{x}_i$, for each pair $\tilde{x}_i, \tilde{x}_j$ with probability $\geq 1 - \delta'$ we have:

$$(1 - \epsilon)\|\tilde{x}_i - \tilde{x}_j\|_2 \leq \|x_i - x_j\|_2 \leq (1 + \epsilon)\|\tilde{x}_i - \tilde{x}_j\|_2.$$  

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$m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$

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$\tilde{x}_1, \ldots, \tilde{x}_n$: original points, $x_1, \ldots, x_n$: compressed points, $\Pi \in \mathbb{R}^{m \times d}$: random projection matrix. $d$: original dimension. $m$: compressed dimension, $\epsilon$: embedding error, $\delta$: embedding failure prob.
Claim: If we choose $\Pi$ with i.i.d. $\mathcal{N}(0, 1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\varepsilon^2}\right)$, letting $x_i = \Pi \bar{x}_i$, for each pair $\bar{x}_i, \bar{x}_j$ with probability $\geq 1 - \delta'$ we have:

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$$m = O\left(\frac{\log(1/\delta')}{\varepsilon^2}\right) = O\left(\frac{\log\left(\frac{n}{2}/\delta\right)}{\varepsilon^2}\right)$$

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Claim: If we choose \( \Pi \) with i.i.d. \( \mathcal{N}(0, 1/m) \) entries and 
\[ m = O \left( \frac{\log(1/\delta')}{\epsilon^2} \right), \]
letting \( \bar{x}_i = \Pi \bar{x}_i \), for each pair \( \bar{x}_i, \bar{x}_j \) with probability \( \geq 1 - \delta' \) we have:
\[
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\]

With what probability are all pairwise distances preserved?

Union bound: With probability \( \geq 1 - \binom{n}{2} \cdot \delta' \) all pairwise distances are preserved.

Apply the claim with \( \delta' = \delta / \binom{n}{2} \). \( \implies \) for \( m = O \left( \frac{\log(1/\delta')}{\epsilon^2} \right) \), all pairwise distances are preserved with probability \( \geq 1 - \delta \).

\[
m = O \left( \frac{\log(1/\delta')}{\epsilon^2} \right) = O \left( \frac{\log(\binom{n}{2}/\delta)}{\epsilon^2} \right) = O \left( \frac{\log(n^2/\delta)}{\epsilon^2} \right)
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With what probability are all pairwise distances preserved?

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Apply the claim with $\delta' = \delta / \binom{n}{2}$. $\implies$ for $m = O\left(\frac{\log(1/\delta')}{{\epsilon}^2}\right)$, all pairwise distances are preserved with probability $\geq 1 - \delta$.

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Yields the JL lemma.