CMPSCI 311: Introduction to Algorithms

Lecture 18: Network Flow and Intractability

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Today

- SEGMENTATION – Network Flow Application
- Reductions

Image Segmentation

- Using an expensive camera and appropriate lenses, you can get a "bokeh" effect on portrait photos in which the background is blurred and the foreground is in focus.

- But using cheap cameras in phones and appropriate software you can fake this effect...

Formulating the problem

- SEGMENTATION or Foreground-Background separation.
- Input:
  - Set S pixels in $n \times n$ grid with set N of neighboring pairs.
  - ith pixel has foreground score $f_i \geq 0$ and background score $b_i \geq 0$.
  - Each $(i,j) \in N$ has penalty $p_{ij} \geq 0$ for labeling one in foreground and one in background.
- Goal: Partition pixels into foreground F and background $B = S \setminus F$ to maximize

$$\text{score}(F,B) = \sum_{i \in F} f_i + \sum_{j \in B} b_j - \sum_{(i,j) \in N : i \in F, j \in B} p_{ij}$$

A min-cut problem?

$$\text{score}(F,B) = \sum_{i \in F} f_i + \sum_{j \in B} b_j - \sum_{(i,j) \in N : i \in F, j \in B} p_{ij}$$

- Ignoring first two terms, almost a min-cut problem!
- Observation: Define

$$\begin{aligned}\text{score}'(F,B) &= \sum_{i \in S} f_i + \sum_{j \in S} b_j - \text{score}(F,B) \\ &= \sum_{i \notin F} f_i + \sum_{j \notin B} b_j + \sum_{(i,j) \in N : i \in F, j \in B} p_{ij}\end{aligned}$$

  - Maximizing score equivalent to minimizing score$'$.
  - Even closer to a min-cut problem!

Turning the problem into a network flow problem

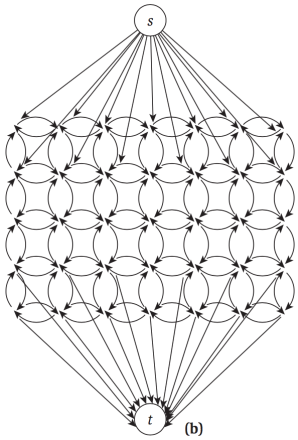
$$\text{score}'(F,B) = \sum_{i \notin F} f_i + \sum_{j \notin B} b_j + \sum_{(i,j) \in N : i \in F, j \in B} p_{ij}$$

- Define the directed graph G where
  - Pixels S, are nodes of G
  - For each $(i,j) \in N$, edge in each direction with capacity p_{ij}
  - Node s with an edge to each pixel i with capacity f_i
  - Node t with an edge from each pixel j with capacity b_j
- Observe:

$$\text{score}'(F,B) \quad = \quad \text{cut}(F,B)$$

- So finding minimum cut in G is equivalent to maximizing the image segmentation score.

Example



Formally...

Claim. $(F^\star, B^\star)$ maximizes $\text{score} \Leftrightarrow (F^\star, B^\star)$ minimizes cut.

- Suppose $(F^\star, B^\star)$ maximizes score. Then for any (F, B).

$$\begin{aligned} 0 &\leq \text{score}(F^\star, B^\star) - \text{score}(F, B) \\ &= \text{score}'(F, B) - \text{score}'(F^\star, B^\star) \\ &= \text{cut}(F, B) - \text{cut}(F^\star, B^\star) \end{aligned}$$

- Hence $F^\star, B^\star$ also minimizes cut.
- Other direction is analogous.
- Note: You have to prove both.

Reductions

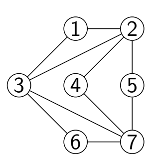
- We just showed how to *reduce* SEGMENTATION to NETWORKFLOW.
  - Given algorithm for NETWORKFLOW (e.g., Ford-Fulkerson) we can easily solve SEGMENTATION.
  - Therefore, segmentation is "no harder" than network flow.
- **Definition:** Problem Y is *poly-time reducible* to problem X if:
  - We can solve Y using polynomially many computations + polynomially many calls to black-box algorithm for X.
  - Or, if we can solve X in polynomial time, we can solve Y in polynomial time as well.
  - Write $Y \leq_P X$.
- SEGMENTATION $\leq_P$ NETWORKFLOW

Reducibility and Intractability

- **Claim 1.** If $Y \leq_P X$ and X poly-time solvable, so is Y.
  - Can use to design algorithms.
- **Claim 2.** If $Y \leq_P X$ and Y *not* poly-time solvable, then X is not either.
  - Contrapositive of above.
  - Can be used to prove hardness.
- The catch: we do not know of any problem Y that provably *cannot* be solved in polynomial time.

A first reduction

Definition. $S \subset V$ is an *independent set* in a graph $G = (V, E)$ if no nodes in S share an edge.
Problem. Does G have independent set of size at least k?



Definition. $S \subset V$ is a *vertex cover* in a graph $G = (V, E)$ if every edge adjacent to some $v \in S$.
Problem. Does G have vertex cover of size at most k?

The reduction

Claim. S is independent if and only if $V \setminus S$ is a vertex cover.
Proof.

- Suppose S independent but $V \setminus S$ is not a vertex cover.
  - Then exists $(u, v) \in E$ with $u, v \notin V \setminus S$.
  - Implies $u, v \in S$, but S independent. Contradiction.
- Suppose $V \setminus S$ is a vertex cover but S is not independent.
  - Then exists $u, v \in S$ with $(u, v) \in E$.
  - But edge (u, v) not covered by $V \setminus S$, contradiction.

Theorem. INDEPENDENTSET $\leq_P$ VERTEXCOVER and VERTEXCOVER $\leq_P$ INDEPENDENTSET.

Reduction #2: Set cover

Problem. Given a set U of n elements, subsets $S_1, \ldots, S_m \subset U$, and a number k, does there exist a collection of at most k subsets S_i whose union is U?

- Example:
  - U is the set of all skills.
  - Each S_i is a person.
  - Want to find a small team that has all skills.
- **Theorem.** $\textsc{VertexCover} \leq_P \textsc{SetCover}$

Set cover reduction

Reduction. Given $G = (V, E)$ make set cover instance with $U = E$, and S_v is all edges incident to v. Keep k the same.
Proof. U covered with at most k sets if and only if E covered by at most k vertices.

- If $v_1, \ldots, v_\ell$ is a VC then $S_{v_1}, \ldots, S_{v_\ell}$ is a SC.
- If $S_{i_1}, \ldots, S_{i_\ell}$ covers U, then every edge adjacent to one of $\{i_1, \ldots, i_\ell\}$.

Interlude

- Decision versus Optimization
  - Algorithms so far have been for optimization
  - Reductions so far have been for decision
- But can reduce optimization to decision and vice versa.
  - e.g., solve $\textsc{MaxIndSet}$(G) by solving $\textsc{IndSet}(G, k)$ for $k = 1, \ldots, n$.
  - e.g., solve $\textsc{IndSet}$(G,k) by computing $S = \textsc{MaxIndSet}(G)$ and output $\mathbf{1}[|S| \geq k]$.

Common Confusions

$Y \leq_P X$ means:

- Y is "no harder" than X
- X is "at least as hard" as Y.
- To show Y is easy, show $Y \leq_P X$ for easy X.
- To show X is hard, show $Y \leq_P X$ for hard Y.

For decision problem Y, need to show two things.

- Correctly outputs $\textsc{Yes}$ and $\textsc{No}$.

A bad reduction.

Given $\textsc{VertexCover}$ instance (G, k), make $\textsc{SetCover}$ instance with $U = E$, S_v is edges incident to v, $S_0 = U$, and integer k.

- If G has VC of size at most k, then U has cover of size at most k.
- But if U has cover of size k, G might not!

If (G, k) is a $\textsc{No}$ instance, the reduction does not correctly return $\textsc{No}$.

Reduction #3: Satisfiability

- Can we determine if a boolean formula has a satisfying assignment?
- Let $X = \{x_1, \ldots, x_n\}$ be boolean variables
  - A term or literal is x_i or $\bar{x}_i$.
  - A clause is *or* of several terms $(t_1 \vee t_2 \vee \ldots \vee t_\ell)$.
  - A formula is *and* of several clauses
  - An assignment $v : X \to \{0, 1\}$ gives T/F to each variable.
- v satisfies formula if all clauses evaluate to True.

Example.

$$(x_1 \vee \bar{x}_2) \wedge (x_1 \vee x_4 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_4) \wedge (x_3 \vee x_2)$$

Reduction #3: Satisfiability

SAT – Given boolean formula $C_1 \wedge C_2 \ldots \wedge C_m$ over variables $X = \{x_1, \ldots, x_n\}$, does there exist a satisfying assignment?

3-SAT – Given boolean formula $C_1 \wedge C_2 \ldots \wedge C_m$ over variables $X = \{x_1, \ldots, x_n\}$ where each C_i has three literals, does there exist a satisfying assignment?

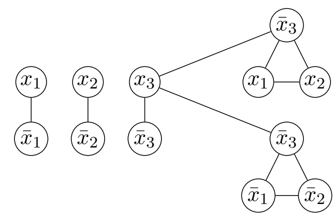
- Any algorithms?

Theorem. $\text{3-SAT} \leq_P \textsc{IndependentSet}$.

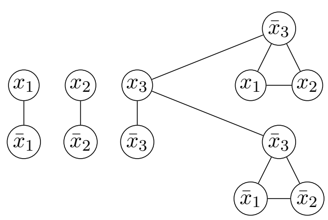
Reduction #3: Satisfiability

$$(x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x_2} \vee \bar{x}_3)$$

- Associate nodes in graph with literals (≥ 2 per variable).
- If $v(x_i) = 1$ in assignment, then cannot select some nodes.
- Associate 3 nodes per clause in a *gadget*.



Satisfiability Proof



Claim Graph has IS of size $n + m$ if and only if formula satisfiable.

- If formula satisfiable, select correct term on the left and one per clause on the right.
- If graph has IS,
  - At most one node per clause on the right
  - At most one node per variable on the left.
  - If node selected in clause, its negation cannot be selected.