CMPSCI 311: Introduction to Algorithms

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Plan

- Review:
  - Breadth First Search
  - Depth First Search
- Traversal Implementation and Running Time
- Traversal Applications
- Directed Graphs

Recall

- Graph $G = (V, E)$
- Set of nodes V of size n
- Set of edges E of size m

Adjacency List Representation

Adjacency List Representation.

- Nodes numbered $1, \ldots, n$.
- `Adj`$[v]$ points to a list of all of v's neighbors.

BFS Description

Define layer L_i = all nodes at distance exactly i from s.

Layers

- $L_0 = \{s\}$
- L_1 = all neighbors of L_0
- L_2 = all nodes with an edge to L_1 that don't belong to L_0 or L_1
- $\ldots$
- L_{i+1} = nodes with an edge to L_i that don't belong to any earlier layer.

$$L_{i+1} = \{v : \exists (u, v) \in E, u \in L_i, v \notin (L_0 \cup \ldots \cup L_i)\}$$

DFS Descriptions

Depth-first search: keep exploring from the most recently discovered node until you have to backtrack.

DFS(u)
  Mark u as "Explored"
  for each edge (u, v) incident to u **do**
    if v is not marked "Explored" **then**
      Recursively invoke DFS(v)
    end if
  end for

Traversal Implementations

Maintain set of explored nodes and discovered

- Explored = have seen this node and explored its outgoing edges
- Discovered = the "frontier". Have seen the node, but not explored its outgoing edges.

Generic Graph Traversal

Let A = data structure of discovered nodes
Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A ▷ w is discovered
      end for
    end if
  end while

Note: one part of this algorithm seems really dumb. Why?
Can put multiple copies of a single node in A.

Generic Graph Traversal

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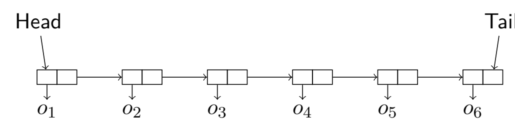
    end if
  end while

BFS: A is a queue (FIFO)
DFS: A is a stack (LIFO)

Interlude (Data Structures)

Linked List:



- Always remove items from front (Head)
- Queue: Insert at Tail (FIFO)
- Stack: Insert at Head (LIFO)
- Insert/Removal are $O(1)$ operations.

BFS Implementation

Let A = empty Queue structure of discovered nodes
Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A ▷ w is discovered
      end for
    end if
  end while

Is this actually BFS? Yes
Running time: $O(n + m)$

BFS Running Time

- Naive $O(n^2)$
- Smarter $O(n + m)$

DFS Implementation

Let $A =$ empty Stack structure of discovered nodes
Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A ▷ w is discovered
      end for
    end if
  end while

Is this actually DFS? Yes
What's the running time?

Back to Connected Components

FindCC(G)
  while There is some unexplored node s **do**
    BFS(s)
    Extract connected component $C(s)$.
  end while

Running time for finding connected components?
Naive: $O(m + n)$ for each component $\Rightarrow O(c(m + n))$ if c components.
Better:

- BFS on component C only works on nodes/edges in C.
- Running time is $O(\sum_C |V(C)| + |E(C)|) = O(m + n)$.

Bipartite Graphs

Definition Graph $G = (V, E)$ is bipartite if V can be partitioned into sets X, Y such that every edge has one end in X and one in Y.

Example Student-College Graph in stable matching
Counter example Cycle of length k for k odd.

Claim If G is bipartite then it cannot contain an odd cycle.

Bipartite Testing

Question Given $G = (V, E)$, is G bipartite?

How do we design an algorithm to test bipartiteness?

- BFS(s) for any s, keep track of layers.
- Nodes in odd layers get color blue, even get color red.
- After, check all edges have different colored endpoints.

Running time? $O(n + m)$.

Analysis of Bipartite Testing

Claim After running BFS on a connected graph G, either,

- There are no edges between two nodes of the same layer $\Rightarrow G$ is bipartite.
- There is an edge between two nodes of the same layer $\Rightarrow G$ has an odd cycle, is not bipartite.

G bipartite if and only if no odd cycles.

Directed Graphs

- Directed Graph $G = (V, E)$.
- V is a set of vertices/nodes.
- E is a set of ordered pairs (u, v).
  - Express asymmetrical relationship

Examples Twitter network, course schedule, web graph.

Adjacency Lists

Maintain two lists.

- `Enter`$[v]$ contains all edges pointing to v.
- `Leave`$[v]$ contains all edges pointing from v.

Strong Connectivity

Definition G is strongly connected if for every $u, v \in V$, there is a path from u to v and from v to u.

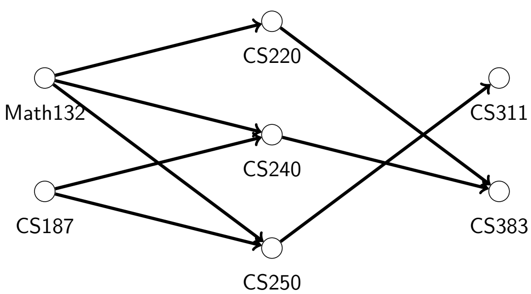
Problem Test if G is strongly connected?

Definition The strongly connected component containing vertex s is the set of all nodes with paths to and from s.

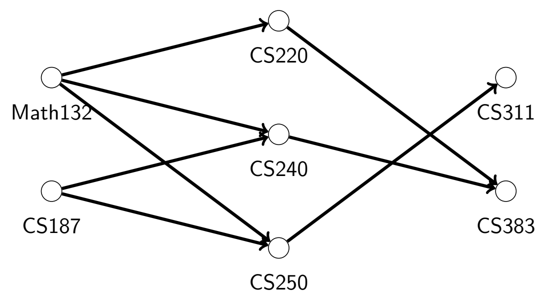
Think about Can you find all SCCs in linear time?

Directed Acyclic Graphs

Definition A directed acyclic graph (DAG) is a directed graph with no cycles.
Example Course prerequisites



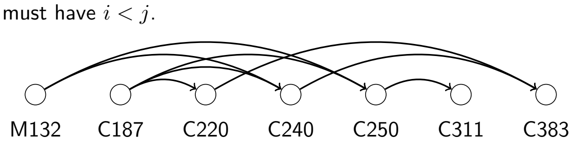
Topological Sorting



Can you find a way to take all of the courses?

Topological Sorting

Definition A topological ordering of $G = (V, E)$ is an ordering $v_1, v_2, \ldots, v_n$ of the nodes, such that for all edges $(v_i, v_j) \in E$, we must have $i < j$.



Claim If G has a topological ordering, then G is a DAG.

Topological sorting

Problem Given DAG G, compute a topological ordering for G.

- Does one always exist?

topo-sort(G)
  while there are nodes remaining **do**
    Find a node v with no incoming edges
    Place v next in the order
    Delete v and all of its outgoing edges from G
  end while

Running time? $O(n^2 + m)$ easy, $O(m + n)$ more clever.

Topological Sorting Analysis

- In a DAG, there is always a node v with no incoming edges.
- Removing a node v from a DAG, produces a new DAG.
- Any node with no incoming edges can be first in topological ordering.

Theorem G is a DAG if and only if G has a topological ordering.

Graphs Summary

- Graph Traversal
  - BFS/DFS, Connected Components, Bipartite Testing
  - Traversal Implementation and Analysis
- Directed Graphs
  - Strong Connectivity
  - Directed Acyclic Graphs
  - Topological ordering