CMPSCI 311: Introduction to Algorithms
Practice Final Exam

Name: ______________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, *unless the problem says otherwise*, you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question	Value	Points Earned
1	10	
2	20	
3	30	
4	10	
5	20	
6	10	
Total	100	

Question 1. (*10 points*) **True of False?** Indicate whether each of the following statements is TRUE or FALSE. No justification required.

1.1 (*2 points*): 3-COLORING *can be solved by breadth first search and therefore is in* P.

1.2 (*2 points*): $\sum_{i=1}^{n} 2^i = \Theta(2^n)$.

1.3 (*2 points*): *A dynamic program that implements the following recursive form can be used to solve the subset sum problem, which asks to find a subset* S *of numbers from* $x_1, \ldots, x_n$ *(all non-negative) with maximum weight subject to not exceeding a given number* W.

$$\mathit{Val}(i, w) = \max\{\, \mathit{Val}(i-1, w), x_i + \mathit{Val}(i-1, w - x_i)\}$$

1.4 (*2 points*): *For any flow network, and any two vertices* s, t *there is always a flow of at least* 1 *from source* s *to target* t.

1.5 (*2 points*): *The recurrence* $T(n) = 2T(n-1) + O(1)$ *solves to* $\Theta(n^2)$.

Question 2. (*20 points*) **Short Answer.** Answer each of the following questions in at most two sentences.

2.1 (*4 points*): *In a weighted graph G where all edges have weight 1, how can we use Djikstra's algorithm to find a minimum spanning tree?*

2.2 (*4 points*): *Solve the recurrence $T(n) = 3T(n/2) + O(n)$.*

2.3 (*4 points*): *Suppose a dynamic programming algorithm creates an $n \times m$ table and to compute each entry of the table it takes a minimum over at most m (previously computed) other entries. What would the running time of this algorithm be, assuming there is no other computations.*

2.4 (*4 points*): *Why is* MaxFlow *in* $\mathcal{NP} \bigcap$ *co-*$\mathcal{NP}$*?*

2.5 (*4 points*): *Suppose* $\mathcal{A}$ *is a randomized algorithm that finds the optimal solution to some minimization problem with probability at least* $p \in (0,1)$*. More precisely, if we run* $\mathcal{A}$ *o some input, it returns a candidate solution* O *along with the cost of* O*, and with probability at least* p*, we are guaranteed that* O *minimizes the cost function. For another parameter* $\delta > 0$*, how can we use* $\mathcal{A}$ *to find an optimal solution with probability at least* $1 - \delta$ *and what is the running time of this new algorithm?*

Question 3. (*20 points*) Consider the longest increasing subsequence problem defined as follows. Given a list of numbers $a_1, \ldots, a_n$ an increasing subsequence is a list of indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $i_1 < i_2 < \ldots, i_k$ and $a_{i_1} \leq a_{i_2} \leq \ldots, \leq a_{i_k}$. The longest increasing subsequence is the longest list of indices with this property.

3.1 (*2 points*): *What is the longest increasing subsequence of the list* $5, 3, 4, 8, 7, 10$*?*

3.2 (*4 points*): *Consider the greedy algorithm that chooses the first element of the list, and then repeatedly chooses the next element that is larger. Is this a correct algorithm? Either prove its correctness or provide a counter example.*

3.3 (*4 points*): *Consider the greedy algorithm that chooses the smallest element of the list, and then repeatedly chooses the smallest element that comes after this chosen one. Is this a correct algorithm? Either prove its correctness or provide a counterexample.*

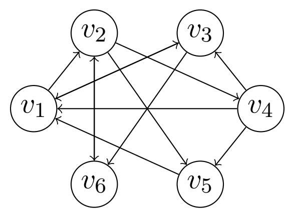
3.4 (*5 points*): *Consider a divide and conquer strategy that splits the list into the first half and second half, recursively computes* $L = (\ell_1, \ldots, \ell_{k_L}), R = (r_1, \ldots, r_{k_R})$ *the longest increasing subsequences in each half, and then, if the last chosen element in the first half is less than the first chosen index in the second half (i.e.* $a_{\ell_{k_L}} \leq a_{r_1}$*) returns* $L \cdot R$*, otherwise it returns the longer of L and R. Is this a correct algorithm? either prove its correctness of provide a counterexample.*

3.5 (*15 points*): *Design a dynamic programming algorithm for longest increasing subsequence. Prove its correctness and analyze its running time.*

Question 4. (*10 points*) In this problem we investigate the feedback arc-set problem which generalizes the topological ordering. Given a directed graph (which may contain cycles), the goal in feedback arc-set is to find an ordering of the vertices that minimizes the number of back edges. More precisely, if $G = (V, E)$ is a directed graph, and $(a_1, \ldots, a_n) \in V$ with $a_i \neq a_j$ is an ordering of the vertices, we define the cost as

$$\mathrm{cost}(a_1, \ldots, a_n) = \sum_{i<j} \mathbf{1}[(a_j, a_i) \in E]$$

Here $\mathbf{1}[\cdot]$ is a function that is 1 if the argument is true and zero otherwise. This is the number of edges going from right to left (backward) if we ordered the vertices with a_1 on the left and a_n on the right.



4.1 (*1 points*): *In the above graph, what is the cost of* $(v_1, v_2, v_3, v_4, v_5, v_6)$*?*

4.2 (*1 points*): *In the above graph, what is the cost of* $(v_6, v_5, v_4, v_3, v_2, v_1)$*?*

4.3 (*2 points*): *True or False. a directed acyclic graph always has an ordering* O *with cost*$(O) = 0$.

4.4 (*6 points*): *Prove that the decision version of feedback arc set is NP-complete. That is given a directed graph and an integer* k*, decide whether the graph has an ordering with at most* k *back-edges.*

Question 5. (*20 points*) In this problem we investigate vertex-capacitated flow networks. We are given a directed graph $G = (V, E)$ with source s and sink t and a capacity c_v for each $v \in V$. We want an $s - t$ flow f that satisfies the usual conservation of flow constraints, but instead of satisfying edge-capacity constraints, satisfies the vertex capacity constraints $f(v) \leq c_v$. Here $f(v) = \sum_{(u,v) \in E} f_{u,v}$ is the total flow entering the node v. The goal is to design an algorithm for computing a maximum $s - t$ flow in a vertex-capacitated network.

5.1 (*5 points*): *Draw a directed graph G with clearly labeled source s and sink t, where if we consider the usual edge-capacitated version of the problem (with edge capacities $c_e = 1$) we get a maximum flow with a different value than if we consider the vertex capacitated version of the problem (with vertex capacities $c_v = 1$).*

5.2 (*15 points*): *Design a polynomial time algorithm for computing the maximum flow in a node-capacitated network. Prove that the algorithm is correct and analyze its running time.*

Additional space.

Question 6. (*10 points*) Consider a variant of the subset sum problem where we are given a set of numbers $x_1, \dots, x_n$ and need to partition them numbers into sets $S_1, \dots, S_K$ such that for each $k \in \{1, \dots, K\}$, $\sum_{i \in S_k} x_i \leq W$ for some target W. The goal is to minimize K, the number of sets in the partition. We will study a simple approximation algorithm for this problem. The algorithm considers the items in order, and forms the first set S_1 by repeatedly adding the numbers $x_1, x_2, \dots$ until the next number would exceed the target W. Then it proceed to construct the next set.

6.1 (*2 points*): *Give an example input where this algorithm does not use the minimum number of sets.*

6.2 (*2 points*): *Derive a lower bound on $K^\star$ the smallest possible number of sets in the partition in terms of the target W and the total weight $X = \sum_{i=1}^{n} x_i$.*

6.3 (*6 points*): *Use this lower bound to prove that this greedy algorithm always produces a number of sets K that is at most $2K^\star$.*