Zero-Sum Games

Definition

A two-player, simultaneous-move, zero-sum game consists of a set of $k$ options for player $A$, a set of options $\ell$ for player $B$, and a $k \times \ell$ payoff matrix $P$. If $A$ is chooses her $i$th option and $B$ chooses his $j$th option then $A$ gets $P_{ij}$ and $B$ gets $-P_{ij}$.

For two-finger Morra, the payoff matrix is

$$
\begin{array}{c|cc}
   & 1 \text{ B Finger} & 2 \text{ B Finger} \\
\hline
1 \text{ A Finger} & +2 & -3 \\
2 \text{ A Finger} & -3 & +4 \\
\end{array}
$$

where best strategy was for players to show 1 finger with probability $7/12$ and two fingers with probability $5/12$. 
Definition

If a player picks one of their options, we call it a **pure strategy**. If they pick a distribution over their options, we call it a **mixed strategy**. If one option is better than the other no matter what the other player does, we say the first strategy **dominates** the second.
Example: Three-finger Morra

- Alice and Bob play a game
- Simultaneously Alice picks $a \in \{1, 2, 3\}$ and Bob picks $b \in \{1, 2, 3\}$
- Bob pays Alice $(a + b)$ if $a + b$ is even
- Alice pays Bob $(a + b)$ if $a + b$ is odd
- The payoff matrix is

<table>
<thead>
<tr>
<th></th>
<th>1 B Finger</th>
<th>2 B Finger</th>
<th>3 B Finger</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 A Finger</td>
<td>+2</td>
<td>−3</td>
<td>+4</td>
</tr>
<tr>
<td>2 A Finger</td>
<td>−3</td>
<td>+4</td>
<td>−5</td>
</tr>
<tr>
<td>3 A Finger</td>
<td>+4</td>
<td>−5</td>
<td>+6</td>
</tr>
</tbody>
</table>
Analysis of Three-finger Morra (1/2)

- Suppose A plays “1 A Finger” with probability $r$, “2 A Finger” with probability $s$, and “3 A Finger” with probability $1 - r - s$.

- If B plays “1 B Finger” then A’s expected reward is
  $$2r - 3s + 4(1 - r - s) = 4 - 2r - 7s$$

- If B plays “2 B Finger” then A’s expected reward is
  $$-3r + 4s - 5(1 - r - s) = -5 + 2r + 9s$$

- If B plays “3 B Finger” then A’s expected reward is
  $$4r - 5s + 6(1 - r - s) = 6 - 2r - 11s$$

- Hence, for $r = 1/4$, $s = 1/2$, A gets expected return at least 0.
Suppose B plays “1 B Finger” with probability $r$, “2 B Finger” with probability $s$, and “3 B Finger” with probability $1 - r - s$

If A plays “1 A Finger” then B’s expected reward is

$$-2r + 3s - 4(1 - r - s) = -4 + 2r + 7s$$

If A plays “2 A Finger” then B’s expected reward is

$$3r - 4s + 5(1 - r - s) = 5 - 2r - 9s$$

If A plays “3 A Finger” then B’s expected reward is

$$-4r + 5s - 6(1 - r - s) = -6 + 2r + 11s$$

Hence, for $r = 1/4$, $s = 1/2$, B gets expected return at least 0

Hence, best strategy for each player is show 1 finger with probability 1/4 and 2 fingers with probability 1/2.
## Prisoner’s Dilemma

- Two prisoners are being held pending trial for a crime they are alleged to have committed. The prosecutor offers each a deal:

  “Give evidence against your partner and you'll go free, unless your partner also confesses. If both confess, both get 5 year sentences. If neither confess, both get 1 year sentences. If you don’t confess but your partner does, you get 10 years!”

- Can represent this as a game but it’s not zero-sum:

```
<table>
<thead>
<tr>
<th></th>
<th>B Confesses</th>
<th>B Stays Mute</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Confesses</td>
<td>−5, −5</td>
<td>0, −10</td>
</tr>
<tr>
<td>A Stays Mute</td>
<td>−10, 0</td>
<td>−1, −1</td>
</tr>
</tbody>
</table>
Nash Equilibrium

Definition

A Nash Equilibrium is a set of strategies for each player where no change by one player alone can improve his outcome.

For the prisoners dilemma the unique Nash Equilibrium is that both prisoners confess.

Theorem (Nash)

*Every game where each player has a finite number of options, has at least one Nash equilibrium.*
Hawks and Doves

- Two birds meet over a piece of food and have to decide whether to act aggressive (hawkish) or passive (dovish)
  - If a hawk meets a dove, the hawk gets the food worth 50 points
  - If two hawks meet they both loose -25 points
  - If two doves meet, they both get 15 points

- Can represent this as:

<table>
<thead>
<tr>
<th></th>
<th>B is a Hawk</th>
<th>B is a Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>A is a Hawk</td>
<td>-25, -25</td>
<td>50, 0</td>
</tr>
<tr>
<td>A is a Dove</td>
<td>0, 50</td>
<td>15, 15</td>
</tr>
</tbody>
</table>

- No Nash Equilibrium where both players play same pure strategies:
  - If A and B are Hawks, both would prefer to switch to Doves
  - If A and B are Doves, both would prefer to switch to Hawks
  - A plays Hawk and B plays Dove is a Nash Equilibria and vice versa
A Mixed Strategy Nash Equilibrium for Hawks and Doves

- Suppose Alice play Hawk with probability $0 < q < 1$ and Bob plays Hawk with probability $0 < p < 1$
- Alice’s expected reward is
  
  $$-25pq + 50q(1 - p) + 15(1 - p)(1 - q) = -60pq + 35q + 15 - 15p$$
  
  $$= q(35 - 60p) + 15 - 15p$$

- When can Alice not improve by changing $q$? When $p = 7/12$.
- Bob’s expected reward is
  
  $$-60pq + 35p + 15 - 15q = p(35 - 60q) + 15 - 15q$$

- When can Bob not improve by changing $p$? When $q = 7/12$.
- Hence $p = q = 7/12$ is only Nash Equilibrium with mixed strategies.