CMPSCI 240: Reasoning about Uncertainty

Lecture 23: More Game Theory

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Last Compiled: April 20, 2017

Outline

Last Time: Zero-Sum Games

Definition

A *two-player, simultaneous-move, zero-sum game* consists of a set of k options for player A, a set of options ℓ for player B, and a $k \times \ell$ *payoff matrix* P. If A is chooses her ith option and B chooses his jth option then A gets P_{ij} and B gets $-P_{ij}$.

For two-finger Morra, the payoff matrix is

	1 B Finger	2 B Finger
1 A Finger	$+2$	-3
2 A Finger	-3	$+4$

where best strategy was for players to show 1 finger with probability $7/12$ and two fingers with probability $5/12$.

Outline

Prisoner's Dilemma

- Two prisoners are being held pending trial for a crime they are alleged to have committed. The prosecutor offers each a deal:

  "Give evidence against your partner and you'll go free, unless your partner also confesses. If both confess, both get 5 year sentences. If neither confess, both get 1 year sentences. If you don't confess but your partner does, you get 10 years!"

- Can represent this as a game but it's not zero-sum:

	B Confesses	B Stays Mute
A Confesses	$-5, -5$	$0, -10$
A Stays Mute	$-10, 0$	$-1, -1$

Nash Equilibrium

Definition

A Nash Equilibrium is a set of strategies for each player where no change by one player alone can improve his outcome.

For the prisoners dilemma the unique Nash Equilibrium is that both prisoners confess.

Theorem (Nash)

Every game where each player has a finite number of options, has at least one Nash equilibrium.

Hawks and Doves

- Two birds meet over a piece of food and have to decide whether to act aggressive (hawkish) or passive (dovish)
  - If a hawk meets a dove, the hawk gets the food worth 50 points
  - If two hawks meet they both loose -25 points
  - If two doves meet, they both get 15 points
- Can represent this as:

	B is a Hawk	B is a Dove
A is a Hawk	$-25, -25$	$50, 0$
A is a Dove	$0, 50$	$15, 15$

- No Nash Equilibrium where both players play same pure strategies:
  - If A and B are Hawks, both would prefer to switch to Doves
  - If A and B are Doves, both would prefer to switch to Hawks
- A plays Hawk and B plays Dove is a Nash Equilibria and vice versa

A Mixed Strategy Nash Equilibrium for Hawks and Doves

- Suppose Alice play Hawk with probability $0 < q < 1$ and Bob plays Hawk with probability $0 < p < 1$
- Alice's expected reward is

$$\begin{aligned} -25pq + 50q(1-p) + 15(1-p)(1-q) &= -60pq + 35q + 15 - 15p \\ &= q(35-60p) + 15 - 15p \end{aligned}$$

- When can Alice not improve by changing q? When $p = 7/12$.
- Bob's expected reward is

$$-60pq + 35p + 15 - 15q = p(35-60q) + 15 - 15q$$

- When can Bob not improve by changing p? When $q = 7/12$.
- Hence $p = q = 7/12$ is only Nash Equilibrium with mixed strategies.

Outline

Iterated Prisoners Dilemma

Suppose you are playing an prisoner dilemma game multiple times with someone, what should you do?

Recall the sentencing matrix:

	B Confesses	B Stays Mute
A Confesses	-5,-5	0, -10
A Stays Mute	-10, 0	-1, -1

Suppose after each game, we play again with probability p. We can model the number of games played as a geometric random variable. Since the stopping condition is the probability of of *not* continuing the game, we can say that the expected number of games is $\frac{1}{1-p}$.

Playing against someone who always confesses

- If you know the other player will always confess, you minimize your losses by always confessing as well.
- Let X be a random variable representing the number of rounds for which the game is played.
- Since you'd lose 5 units every round, your expected payoff is

$$-5 \times E(X) = -\frac{5}{1-p}$$

Playing against someone who retaliates

- Suppose you know your opponent will confess on every turn once you have confessed once, but will stay mute until then. For what values of p should you confess on the first turn?
- Let X be a random variable representing the number of rounds for which the game is played.
- If you confess from the first round onwards your payoff is:

$$0 + (-5) \times (X - 1)$$

  which is $-5/(1-p) + 5$.
- If you never confess, your payoff is $-1/(1-p)$.
- Hence you should confess from the start if

$$-5/(1-p) + 5 \geq -1/(1-p)$$

  i.e., $p \leq 1/5$.