CMPSCI 240: Reasoning about Uncertainty

Lecture 21: Game Theory

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Outline

1. Game Theory

2. Non Zero-Sum Games and Nash Equilibrium
Example: Two-finger Morra

- Alice and Bob play a game
- Simultaneously Alice picks $a \in \{1, 2\}$ and Bob picks $b \in \{1, 2\}$
- Bob pays Alice $(a + b)$ if $a + b$ is even
- Alice pays Bob $(a + b)$ if $a + b$ is odd

Obviously if Bob always plays the same number, Alice can take advantage of this. What if Bob plays different numbers with different probabilities?
Analysis of Two-finger Morra (1/3)

- Suppose Bob plays “1” with prob. $q$ and “2” with prob. $1 - q$
- If Alice plays “1” then Bob has expected return
  
  $$-2q + 3(1 - q) = -5q + 3$$

- If Alice plays “2” then Bob has expected return
  
  $$3q - 4(1 - q) = -4 + 7q$$

- Then no matter what Alice does, Bob expects to get
  
  $$\min(-5q + 3, -4 + 7q)$$

- If Bob sets $q = 7/12$ his expected earning is at least:
  
  $$\min(-5 \times 7/12 + 3, -4 + 7 \times 7/12) = 1/12$$
Analysis of Two-finger Morra (2/3)

- Suppose Alice plays “1” with prob. $p$ and “2” with prob. $1 - p$
- If Bob plays “1” then Alice expects to get
  \[2p - 3(1 - p) = -3 + 5p\]
- If Bob plays “2” then Alice expects to get
  \[-3p + 4(1 - p) = -7p + 4\]
- Then no matter what Bob does, Alice expects to get
  \[\min(-3 + 5p, -7p + 4)\]
- If Alice sets $p = 7/12$ her expected earning is at least:
  \[\min(-3 + 5 \times 7/12, -7 \times 7/12 + 4) = -1/12\]
Conclusion:

- Alice’s expected earning plus Bob’s expected earnings always sum up to 0.
- Bob can ensure his expected earning is at least $\frac{1}{12}$; so Alice’s expected earning is at best $-\frac{1}{12}$.
- Alice can ensure his expected earning is at least $-\frac{1}{12}$; so Bob’s expected earning is at best $\frac{1}{12}$.
- Hence the strategies that ensured Bob got $\frac{1}{12}$ in expectation and Alice got $-\frac{1}{12}$ in expectation are optimal, i.e., both players play 1 finger with probability $\frac{7}{12}$ and 2 fingers with probability $\frac{5}{12}$. 
Zero-Sum Games

Definition

A two-player, simultaneous-move, zero-sum game consists of a set of \( k \) options for player A, a set of options \( \ell \) for player B, and a \( k \times \ell \) payoff matrix \( P \). If A is chooses her \( i \)th option and B chooses his \( j \)th option then A gets \( P_{ij} \) and B gets \(-P_{ij}\).

For two-finger Morra, the payoff matrix is

\[
\begin{array}{c|cc}
 & 1 \text{ B Finger} & 2 \text{ B Finger} \\
\hline
1 \text{ A Finger} & +2 & -3 \\
2 \text{ A Finger} & -3 & +4 \\
\end{array}
\]

where best strategy was for players to show 1 finger with probability \(7/12\) and two fingers with probability \(5/12\).
Pure and Mixed Strategies

**Definition**

If a player picks one of their options, we call it a pure strategy. If they pick a distribution over their options, we call it a mixed strategy. If one option is better than the other no matter what the other player does, we say the first strategy dominates the second.
Example: Two-finger Morra with Slightly Different Pay-off

<table>
<thead>
<tr>
<th></th>
<th>1 B Finger</th>
<th>2 B Finger</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 A Finger</td>
<td>+2</td>
<td>−3</td>
</tr>
<tr>
<td>2 A Finger</td>
<td>−2</td>
<td>+4</td>
</tr>
</tbody>
</table>

- If Alice plays “1” with prob. \( p \) and “2” with prob. \( 1 - p \) then her expected payoff is at least:

\[
\min(2p - 2(1 - p), -3p + 4(1 - p))
\]

which is at least \( \frac{2}{11} \) when \( p = \frac{6}{11} \).

- If Bob plays “1” with prob. \( q \) and “2” with prob. \( 1 - q \) then his expected payoff is at least:

\[
\min(-2q + 3(1 - q), 2q - 4(1 - q))
\]

which is at least \( -\frac{2}{11} \) when \( q = \frac{7}{11} \).

- Hence, Alice’s best strategy is one finger with probability \( \frac{6}{11} \) and Bob’s best strategy is one finger with probability \( \frac{7}{11} \).
Example: Three-finger Morra

- Alice and Bob play a game
- Simultaneously Alice picks $a \in \{1, 2, 3\}$ and Bob picks $b \in \{1, 2, 3\}$
- Bob pays Alice $(a + b)$ if $a + b$ is even
- Alice pays Bob $(a + b)$ if $a + b$ is odd
- The payoff matrix is

<table>
<thead>
<tr>
<th></th>
<th>1 B Finger</th>
<th>2 B Finger</th>
<th>3 B Finger</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 A Finger</td>
<td>+2</td>
<td>−3</td>
<td>+4</td>
</tr>
<tr>
<td>2 A Finger</td>
<td>−3</td>
<td>+4</td>
<td>−5</td>
</tr>
<tr>
<td>3 A Finger</td>
<td>+4</td>
<td>−5</td>
<td>+6</td>
</tr>
</tbody>
</table>
Analysis of Three-finger Morra (1/2)

- Suppose A plays “1” with probability \( r \), “2” with probability \( s \), and “3” with probability \( 1 - r - s \).
- If B plays “1” then A’s expected reward is
  \[
  2r - 3s + 4(1 - r - s) = 4 - 2r - 7s
  \]
- If B plays “2” then A’s expected reward is
  \[
  -3r + 4s - 5(1 - r - s) = -5 + 2r + 9s
  \]
- If B plays “3” then A’s expected reward is
  \[
  4r - 5s + 6(1 - r - s) = 6 - 2r - 11s
  \]
- Hence, for \( r = 1/4, s = 1/2 \), A gets expected return at least 0.
Analysis of Three-finger Morra (2/2)

- Suppose B plays “1” with probability $r$, “2” with probability $s$, and “3” with probability $1 - r - s$

- If A plays “1” then B’s expected reward is
  
  $$-2r + 3s - 4(1 - r - s) = -4 + 2r + 7s$$

- If A plays “2” then B’s expected reward is
  
  $$3r - 4s + 5(1 - r - s) = 5 - 2r - 9s$$

- If A plays “3” then B’s expected reward is
  
  $$-4r + 5s - 6(1 - r - s) = -6 + 2r + 11s$$

- Hence, for $r = 1/4$, $s = 1/2$, B gets expected return at least 0

Hence, best strategy for each player is show 1 finger with probability 1/4 and 2 fingers with probability 1/2.
Outline

1. Game Theory

2. Non Zero-Sum Games and Nash Equilibrium
**Prisoner’s Dilemma**

- Two prisoners are being held pending trial for a crime they are alleged to have committed. The prosecutor offers each a deal:

  "Give evidence against your partner and you'll go free, unless your partner also confesses. If both confess, both get 5 year sentences. If neither confess, both get 1 year sentences. If you don’t confess but your partner does, you get 10 years!"

- Can represent this as a game but it’s not zero-sum:

<table>
<thead>
<tr>
<th></th>
<th>B Confesses</th>
<th>B Stays Mute</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Confesses</td>
<td>−5, −5</td>
<td>0, −10</td>
</tr>
<tr>
<td>A Stays Mute</td>
<td>−10, 0</td>
<td>−1, −1</td>
</tr>
</tbody>
</table>

In each entry, the first number is A's reward and the second number of B’s reward.
Nash Equilibrium

**Definition**

A Nash Equilibrium is a set of strategies for each player where no change by one player alone can improve his outcome.

For the prisoners dilemma the unique Nash Equilibrium is that both prisoners confess.

**Theorem (Nash)**

*Every game where each player has a finite number of options, has at least one Nash equilibrium.*