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2 Partitions

3 More Examples

4 Clicker Questions

5 Bonus: Coin Flips

6 Bonus: A Harder Counting Problem
Shortcuts for Counting

- **Permutations**: There are \( n! = n \times (n-1) \times \ldots \times 2 \times 1 \) ways to permute \( n \) objects. E.g., permutations of \( \{a, b, c\} \) are

  \[ abc, acb, bac, bca, cab, cba \]

- **\( k \)-Permutations**: There are \( n \times (n-1) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!} \) ways to choose the first \( k \) elements of a permutation of \( n \) objects. E.g., 2-permutations of \( \{a, b, c, d\} \) are

  \[ ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc \]

- **Combinations**: There are \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) ways to choose a subset of size \( k \) from a set of \( n \) objects. E.g., the subsets of \( \{a, b, c, d\} \) of size 2 are

  \[ \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \]
Counting Combinations

Let $S$ be a set of $n$ objects. How many subsets of size $k$ are there?

Every subset corresponds to $k!$ different $k$-permutations, so the number of “$k$-combinations” is

$$\frac{\text{the number of } k\text{-permutations}}{k!} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \frac{n!}{(n-k)!k!}$$

which is denoted $\binom{n}{k}$, pronounced “$n$ choose $k$”.
Counting Sequences

Example: Let $S = \{a, b, c, d\}$ be a set of size $n = 4$ and consider subsets of size $k = 2$

<table>
<thead>
<tr>
<th>Subsets of size 2</th>
<th>2-permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a,b}</td>
<td>ab, ba</td>
</tr>
<tr>
<td>{a,c}</td>
<td>ac, ca</td>
</tr>
<tr>
<td>{a,d}</td>
<td>ad, da</td>
</tr>
<tr>
<td>{b,c}</td>
<td>bc, cb</td>
</tr>
<tr>
<td>{b,d}</td>
<td>bd, db</td>
</tr>
<tr>
<td>{c,d}</td>
<td>cd, dc</td>
</tr>
</tbody>
</table>

We know there are $n!/(n-k)! = 12$ different 2-permutations and these can be arranged into groups of size $k! = 2$ such that each 2-permutation in the same group corresponds to the same subset of size 2. Hence, the total number of groups is $n!/(n-k)! \times \frac{1}{k!} = \binom{4}{2} = 6$. 
Question: The Flaming Wok restaurant sells a 3-item lunch combo. You can choose from 10 different items. How many different lunch combos are there?

Answer: Since your lunch is the same regardless of the order the items are put on the plate, this is a combination problem. The number of lunch combos is thus \( \binom{10}{3} \) or 120.
Example: Flaming Wok

Question: Suppose you ask for a random combo and there's one item you don't like. What's the combo you order has items you like?

Answer: The probability that you will like a random combo is the number of combos you like, which is \( \binom{9}{3} = 84 \), divided by the number of combos, which is \( \binom{10}{3} \). This gives \( 84/120 = 0.7 \).
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Counting Partitions

- Consider an experiment where we divide \( r \) objects into \( l \) groups with sizes \( k_1, k_2, \ldots, k_l \) such that \( r = \sum_{i=1}^{l} k_i \). The order of items within each group doesn't matter.

- A combination divides items into one group of \( k \) and one group of \( r - k \) and is thus a 2-partition.

- How many partitions are there?

- There are \( \binom{r}{k_1} \) ways to choose the objects for the first group. This leaves \( r - k_1 \) objects. There are \( \binom{r-k_1}{k_2} \) ways to choose objects for the second group. There are \( \binom{r-k_1-k_2-\ldots-k_{l-1}}{k_l} \) ways to choose the objects for the last group.
Counting Partitions

- Using the counting principle, the number of partitions is thus:

\[
\binom{r}{k_1} \cdot \binom{r - k_1}{k_2} \cdots \binom{r - k_1 - k_2 - \ldots - k_{l-1}}{k_l}
\]

\[
= \frac{r!}{k_1!(r - k_1)!} \cdot \frac{(r - k_1)!}{k_2!(r - k_1 - k_2)!} \cdots \frac{(r - k_1 - k_2 - \ldots - k_{l-1})!}{k_l!(r - k_1 - k_2 - \ldots - k_l)!}
\]

- Canceling terms yields the final result:

\[
\frac{r!}{k_1! \cdots k_l!}
\]
Example: Discussion Groups

- **Question:** How many ways are there to split a discussion section of 12 students into 3 groups of 4 students each?

- **Answer:** This is a partition problem with 3 partitions of 4 objects each and 12 objects total. Using the partition counting formula, the answer is:

\[
\frac{12!}{4! \cdot 4! \cdot 4!} = \frac{12!}{(4!)^3} = 34,650
\]
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# Summary of Counting Problems

<table>
<thead>
<tr>
<th>Structure</th>
<th>Description</th>
<th>Order Matters</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutation</td>
<td>Number of ways to order $n$ objects</td>
<td>Yes</td>
<td>$n!$</td>
</tr>
<tr>
<td>$k$-Permutation</td>
<td>Number of ways to form a sequence of size $k$ using $k$ different objects from a set of $n$ objects</td>
<td>Yes</td>
<td>$\frac{n!}{(n-k)!}$</td>
</tr>
<tr>
<td>Combination</td>
<td>Number of ways to form a set of size $k$ using $k$ different objects from a set of $n$ objects</td>
<td>No</td>
<td>$\frac{n!}{k!(n-k)!}$</td>
</tr>
<tr>
<td>Partition</td>
<td>Number of ways to partition $n$ objects into $l$ groups of size $k_1, \ldots, k_l$</td>
<td>No</td>
<td>$\frac{n!}{k_1! \cdots k_l!}$</td>
</tr>
</tbody>
</table>
Example: Grade Assignments

- **Question:** Suppose a professor decides at the beginning of the semester that in a class of 10 students, 3 A’s, 4 B’s, 2 C’s and one C- will be given. How many different ways can the students be assigned grades at the end of the semester?

- **Answer:** This is a partition problem. There are 10 objects and 4 groups. The group sizes are 3, 4, 2, 1. The answer is thus:

  \[
  \frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!}
  \]
Example: Top of the class

**Question:** A computer science program is considering offering three senior year scholarships to their top three incoming seniors worth $10,000, $5,000 and $2,000. If there are 100 incoming seniors, how many ways are there for the scholarships to be awarded?

**Answer:** This is a k-permutation problem. There are 100 students and 3 distinct scholarships. The number of assignments of students to scholarships is thus:

\[
\frac{100!}{(100-3)!} = 100 \cdot 99 \cdot 98
\]
Example: Binary Strings

- **Question:** How many length five binary strings are there with exactly two ones? E.g., 00011, 00101, 00110

- **Answer:** This is a combination problem in disguise! Consider numbering the positions of a binary string \( \{1, 2, 3, 4, 5\} \). If you pick a subset of these of size two and set the corresponding bits to one and the other bits to zero, you get a binary string with exactly two ones. Then the number of strings is

\[
\binom{5}{2} = \frac{5!}{3! \times 2!} = 10
\]
Example: Overbooked

- **Question:** Suppose a class with 50 students is scheduled in a room with only 40 seats. How many ways are there for 40 of the 50 students to get a seat.

- **Answer:** This is a combination problem. We only care if a student gets a seat, not which seat they get. The answer is thus:

\[
\binom{50}{40} = \frac{50!}{40!10!}
\]
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Suppose you bank insists you use a four digit PIN code. How many possible codes are there:

- A: $10!/4!$
- B: $4^{10}$
- C: $\binom{10}{4}$
- D: $10^4$
- E: $\binom{4}{10}$

Correct answer is D: Consider a 4 stage counting process with 10 choices at each stage.
Suppose you bank insists you use a four digit PIN code where every number is odd. How many possible codes are there:

- \( A : 1000 \)
- \( B : 5^{10} \)
- \( C : 5^4 \)
- \( D : \binom{5}{4} \)
- \( E : 10!/5! \)

Correct answer is \( C \): Consider a 4 stage counting process with 5 choices at each stage.
Suppose you bank insists you use a four digit PIN code where every number is different from the one before. How many possible codes are there:

- **A**: $10 \times 9^3$
- **B**: $10^9$
- **C**: $9^4$
- **D**: $\binom{10}{4} \times 9 \times 9 \times 9$
- **E**: $10! / 9!$

Correct answer is **A**: Consider a 4 stage counting process with 10 choices at the first stage and 9 choices are each subsequent stage.
Suppose you bank insists you use a four digit PIN code where all the numbers are distinct. How many possible codes are there:

\[ A : 10 \times 9^3 \quad B : 9^4 \quad C : \binom{10}{4} \quad D : 1 \quad E : 10 \times 9 \times 8 \times 7 \]

Correct answer is \( E \): This is the number of 4-permutations of \( \{0, 1, \ldots, 9\} \).
Suppose you bank insists you use a four digit PIN code that are palindromes, i.e., are the same read forward and backward, e.g., 1221. How many possible codes are there:

- A: 45
- B: 89
- C: 90
- D: 100
- E: 120

Correct answer is D: Consider the 2-stage counting process with 10 choices at both stages that specifies the first two values.
Suppose you bank insists you use a four digit PIN code where every digit is strictly bigger than the previous digit. How many possible codes are there:

\[ A : \binom{10}{4} \quad B : 10 \times 9 \times 8 \times 7 \quad C : 100 \quad D : 10 \times 9^3 \quad E : \frac{10!}{6!} \]

Correct answer is A: Any set of four numbers could appear in your PIN and once you’ve chosen your set of four numbers, only one ordering of these numbers is valid.
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Example: Independent Coin Flips

- Suppose we have a biased coin that lands heads with probability $p$ and tails with probability $(1 - p)$.
- In the next section of the talk, we'll show that if we toss the coin $n$ times then the probability that it lands heads $k$ times is

$$P_n(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$
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Example: Coins

- The commonly used coins in the USA are 1, 5, 10, and 25 cents.
- In a multiset, elements can be repeated, e.g., \{1, 1, 5, 10\}.
- How many multisets of 7 coins are there? E.g., \{1, 1, 1, 1, 1, 1, 1\} or \{1, 1, 1, 5, 10, 25\} . . .
- **General Problem:** In how many ways may we choose \(k\) elements out of a set of size \(n\) if we’re allowed to repeat elements?
- **Answer:** \(\binom{n-1+k}{n-1}\).
Definition

A function $f$ defined on domain $A$ with range $B$ maps each $a \in A$ to exactly one element $f(a) \in B$.

Definition

A function $f : A \to B$ is bijective if for every $b \in B$ there exists a unique $a$ such that $f(a) = b$.

Lemma

If there exists a bijection between two sets $A$ and $B$ then $|A| = |B|$.

One way to show $f$ is a bijection is to find an inverse function $g$ such that for any $b \in B$,

$$g(b) = a \iff f(a) = b$$
Let $S$ be the set of size $k$ subsets of $\{a_1, \ldots, a_n\}$ where elements are chosen with repetition.

Define a function $f : S \to T$ where $T$ is the set of binary strings of length $n - 1 + k$ with $n - 1$ ones. For a set $s \in S$

$$f(s) = 0\ldots01\ldots01\ldots10\ldots0$$

where $\ell_i$ is the number of copies of $a_i$ in $s \in S$.

$f$ is a bijection because there exists an inverse $g$ defined by

$$g(t) = \bigcup_i \{m_i \text{ copies of } a_i\}$$

where $m_i$ is the number of 0's between $(i-1)$th and $i$th 1 in $t \in T$.

Since $f$ is a bijection we know $|S| = |T| = \binom{n+k-1}{n-1}$
**Example**

Suppose $S = \{a, b, c\}$ and we want to choose 4 elements and we’re allowed to pick elements multiple times.

<table>
<thead>
<tr>
<th>Sequence $f(s)$</th>
<th>Multiset of four items $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>000011</td>
<td>{a,a,a,a}</td>
</tr>
<tr>
<td>000101</td>
<td>{a,a,a,b}</td>
</tr>
<tr>
<td>000110</td>
<td>{a,a,a,c}</td>
</tr>
<tr>
<td>001001</td>
<td>{a,a,b,b}</td>
</tr>
<tr>
<td>001010</td>
<td>{a,a,b,c}</td>
</tr>
<tr>
<td>001100</td>
<td>{a,a,c,c}</td>
</tr>
<tr>
<td>010001</td>
<td>{a,b,b,b}</td>
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<td>010010</td>
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<td>{b,b,c,c}</td>
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<tr>
<td>101000</td>
<td>{b,c,c,c}</td>
</tr>
<tr>
<td>110000</td>
<td>{c,c,c,c}</td>
</tr>
</tbody>
</table>
Example: Coins

- The commonly used coins in the USA are 1, 5, 10, and 25 cents.
- How many multisets of 7 coins are there?

\[
\binom{n-1+k}{n-1} = \binom{4-1+7}{4-1} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120
\]

where \( k \) is the number of coins being chosen and \( n \) is the number of different types.
Suppose you bank insists you use a four digit PIN code where every digit is at least as large as the previous digit. How many possible codes are there:

\[ A : \binom{10}{9} \quad B : 10 \times 9 \times 8 \times 7 \quad C : \binom{10}{4} \quad D : \binom{13}{9} \quad E : \binom{13}{10} \]

Correct answer is D: Any set multiset of four numbers could appear in your PIN and once you’ve chosen these, only one ordering of these numbers is valid. There are \( \binom{10-1+4}{10-1} \) such multisets.