CMPSCI 240: Reasoning about Uncertainty
Lecture 5: Total Probability and Independence

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Outline

1. Total Probability and Bayes Theorem
2. Independence
**Total Probability and Bayes Theorem**

- **Total Probability** If $A_1, \ldots, A_n$ partition $\Omega$ then for any event $B$:

  $$P(B) = P(B | A_1)P(A_1) + \ldots + P(B | A_n)P(A_n) = \sum_{i=1}^{n} P(B | A_i)P(A_i)$$

- **Bayes Theorem** If $A_1, \ldots, A_n$ partition $\Omega$ then for any event $B$:

  $$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B | A_i)P(A_i)}{\sum_{j=1}^{n} P(B | A_j)P(A_j)}$$
Example: Taking the bus

Every morning you leave your house and take the first bus that goes to the university. There’s a 25% chance that the first bus that comes will be a red bus and a 75% chance it will be a blue. If you take the red bus, you get to class late 20% of the time. If you take the blue bus, you get to class late 55% of the time. What’s the probability that you get to class late?

**Question:** What events are specified in the problem?

**Answer:**

\[ B_{red} = \text{“red bus is first”}, \quad B_{blue} = \text{“blue bus is first”}, \quad L = \text{“get to class late”} \]

**Question:** What probabilities are specified in the problem?

**Answer:**

\[
P(B_{red}) = 0.25, \quad P(B_{blue}) = 0.75, \quad P(L|B_{red}) = 0.2, \quad P(L|B_{blue}) = 0.55.
\]

Need to compute \( P(L) \): Since \( B_{blue} \) and \( B_{red} \) partition \( \Omega \):

\[
P(L) = P(L|B_{blue})P(B_{blue}) + P(L|B_{red})P(B_{red}) = 0.4625
\]
Example: Taking the bus 2

As before,

\[ P(B_{\text{red}}) = 0.25 , \quad P(B_{\text{blue}}) = 0.75 \]

\[ P(L|B_{\text{red}}) = 0.2 , \quad P(L|B_{\text{blue}}) = 0.55 . \]

Suppose the lecturer observes that you are late. What’s the probability you caught the blue bus?

Need to compute \( P(B_{\text{blue}}|L) \):

\[
P(B_{\text{blue}}|L) = \frac{P(B_{\text{blue}} \cap L)}{P(L)} = \frac{P(L|B_{\text{blue}})P(B_{\text{blue}})}{P(L)} = 0.891891891 \ldots
\]
Example: Testing Stressed Students

Suppose that 1/5 of students are stressed when doing online quizzes. A faculty member at UMass develops a system for recognizing stressed students during these quizzes. The test can correctly identify positive cases 5/6 of the time and correctly identify negative cases 3/4 of the time. What’s the probability that a student is recognized as stressed?

- **Events:** $S =$ “Stressed” and $T =$ “Test positive”.
- **Relationships:** $S$ and $S^C$ partition $\Omega$.
- **Probabilities:** $P(S) = 1/5$, $P(T|S) = 5/6$, $P(T^C|S^C) = 3/4$.
- **Question:** What is $P(T)$?
- **Answer:**

\[
P(T) = P(T|S)P(S) + P(T|S^C)P(S^C) = 5/6 \cdot 1/5 + 1/4 \cdot 4/5 = 11/30.
\]
Example: Testing Stressed Students 2

As before, \( P(S) = \frac{1}{5} \), \( P(T|S) = \frac{5}{6} \), \( P(T^C|S^C) = \frac{3}{4} \) and we've deduced that \( P(T) = \frac{11}{30} \). What's the probability that a student is stressed given that the test is positive?

■ **Question:** What is \( P(S|T) \)?

■ **Answer:**

\[
P(S|T) = \frac{P(S \cap T)}{P(T)} = \frac{1/6}{11/30} = \frac{5}{11}.
\]
Outline

1. Total Probability and Bayes Theorem
2. Independence
Example: Flipping Two Coins

- Consider flipping a fair coin twice in a row.
- If we know the coin is fair, does knowing the result of the first flip give us any information about the result of the second flip?
- What’s the probability the coin comes up heads on the second flip?
- What’s the probability the coin comes up heads on the second flip given that it came up heads on the first flip?
Intuitively, when knowing that one event occurred doesn’t change the probability that another event occurred or will occur, we say that the two events are probabilistically independent.

We say that two events $A$ and $B$ are independent

$$P(A \cap B) = P(A)P(B).$$

and this implies that $P(A|B) = P(A)$ and $P(B|A) = P(B)$ assuming $0 < P(A) < 1$ and $0 < P(B) < 1$. 

**Question:** Suppose you roll two fair four sided dice. Is the event $A =$ “first roll is 3” independent of the event $B =$ “second roll is 4”?

**Answer 1:** Intuitively, like the coin flip, the two rolls have nothing to do with each other so the events $A$ and $B$ should be independent.

**Answer 2:** Formally, $P(A \cap B) = 1/16$ since there are 16 possible outcomes and the event $A \cap B$ refers to exactly one of them. $P(A) = 1/4$ since there’s a 1/4 chance that the first roll is a 3. Similarly, $P(B) = 1/4$. Thus,

$$P(A)P(B) = (1/4)(1/4) = 1/16$$

so the events are independent.
Question: Suppose you roll two fair four sided dice. Are the events $A =$ “maximum is less than 3” and $B =$ “sum is greater than 3” independent?

Answer 1: Intuitively, the answer is no. If the maximum was low it would appear that this should reduce the probability of the sum being greater than 3.
Question: Suppose you roll two fair four sided dice. Are the events $A =$ “maximum is less than 3” and $B =$ “sum is greater than 3” independent?

Answer 2: Formally,

\[ P(A \cap B) = 1/16, \quad P(A) = 1/4 \quad \text{and} \quad P(B) = 13/16. \]

Since $1/16 \neq 1/4 \cdot 13/16$, the events are not independent.
An Event and Its Complement

- **Question:** Are $A$ and $A^c$ independent if $0 < P(A) < 1$?

- **Answer 1:** Intuitively, no. If you know $A$ happens, then you know $A^c$ does not happen.

- **Answer 2:** Formally, $P(A \cap A^c) = P(\emptyset) = 0$. If $0 < P(A) < 1$, then

  \[ P(A)P(A^c) \neq 0. \]
Independence of Three Events

- Three events $A$, $B$, and $C$ are independent if and only if:

\[
P(A \cap B) = P(A) P(B) \\
P(A \cap C) = P(A) P(C) \\
P(B \cap C) = P(B) P(C) \\
P(A \cap B \cap C) = P(A) P(B) P(C)
\]

- Note that pairwise independence does not imply independence.

- Suppose we have a finite collection of events $\mathcal{A} = \{A_1, \ldots, A_N\}$. The events in $\mathcal{A}$ are said to be independent if and only if for any subset $\mathcal{B} \subseteq \mathcal{A}$ containing two or more events we have:

\[
P(\cap_{B \in \mathcal{B}} B) = \prod_{B \in \mathcal{B}} P(B)
\]
Conditional Independence

- \( A \) and \( B \) are **conditionally independent** given \( C \) if and only if
  \[
P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)
  \]

- If \( P(B \mid C) > 0 \) this is equivalent to \( P(A \mid B \cap C) = P(A \mid C) \)
- If \( P(A \mid C) > 0 \) this is equivalent to \( P(B \mid A \cap C) = P(B \mid C) \)
Conditional Independence Example

- I have one fair coin and one biased coin that lands heads with probability $2/3$.
- I pick a coin with equal probability: let $F$ be the event it’s the fair coin and let $F^c$ be the event it’s the biased coin.
- I toss the chosen coin twice: let $A$ be the event the first toss is heads and let $B$ be the event the second toss is heads.
- Are $A$ and $B$ independent? No.

\[
P(A) = P(A|F)P(F) + P(A|F^c)P(F^c) = \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{7}{12} = P(B)
\]

\[
P(A\cap B) = P(A\cap B|F)P(F) + P(A\cap B|F^c)P(F^c) = \frac{1}{4} \times \frac{1}{2} + \frac{4}{9} \times \frac{1}{2} = \frac{25}{72}
\]

- Are $A$ and $B$ independent conditioned on $F$? Yes.

\[
P(A|F) = P(B|F) = 1/2 \quad \text{and} \quad P(A \cap B|F) = 1/4 .
\]