CMPSCI 240: Homework 2

Due: 8pm, Mar. 6, 2017.
To get full marks show your working and write your answer in the simplest form possible.

Question 1. (*8 points*) For each of the following questions you need to show your working to get full marks.

1.1 (*2 points*): *If X is a Geometric random variable with parameter p, what value of $k \in \{1, 2, \ldots\}$ maximizes $P(X = k) = (1-p)^{k-1}p$? This is referred to as the* mode *of the distribution.*

1.2 (*2 points*): *If X is a Poisson random variable with parameter λ, what value of $k \in \{0, 1, 2, \ldots\}$ maximizes $P(X = k)$?*

1.3 (*2 points*): *If X is a Binomial random variable with parameters n and p, what value of $k \in \{0, 1, 2, \ldots, n\}$ maximizes $P(X = k)$?*

1.4 (*2 points*): *If X is a Geometric random variable with parameter p, what is the value of k such that $P(X \leq k)$ and $P(X \geq k)$ are both at least $1/2$. This is referred to as the* median *of the distribution. Hint: You might want to do Question 2.1 before doing this question.*

Question 2. (*8 points*) Let X is a Geometric random variable with parameter p. You answers for the following questions with be a function of p. You should simplify your answer as much as possible. Hint: In parts of this question you might want to use the formula that $1 + x + x^2 + x^3 + \ldots = 1/(1-x)$ if $0 < x < 1$.

2.1 (*2 points*): *What is the value of $P(X \geq k)$?*

2.2 (*2 points*): *What is the value of $P(X \geq 3$ and $X \leq 5)$?*

2.3 (*2 points*): *What is the value of $P(X = 5 | X \geq 3)$?*

2.4 (*2 points*): *Let A be the event that X is a multiple of three. What is the value of $P(A)$?*

Question 3. (*10 points*) Let X be an arbitrary random variable and suppose $Y = aX + b$ for some real numbers a and b.

3.1 (*2 points*): *Prove that $E(Y) = a \times E(X) + b$ if $Y = aX + b$ for real numbers a and b.*

3.2 (*2 points*): *Prove that $var(Y) = a^2 \times var(X)$ if $Y = aX + b$.*

3.3 (*2 points*): *Prove that $var(X) = E(X^2) - (E(X))^2$. Use this to show that $E(X^2) \geq E(X)^2$ for any random variable X.*

We say a discrete random variable is constant if $X(o)$ takes the same value for all $o \in \Omega$. For this question you may assume that probability of any atomic event is strictly positive.

3.4 (*2 points*): *Prove that if a random variable X is constant then $var(X) = 0$.*

3.5 (*2 points*): *Prove that if $var(X) = 0$ then the random variable X is constant.*

Question 4. (*10 points*)

A monkey is sitting at a simplified keyboard that only includes the keys "a", "b", and "c". The monkey presses the keys at random. Let X be the number of keys pressed until the monkey has pressed all the different keys at least once. For example, if the monkey typed "accaacbcaaac..." then X would equal 7 whereas if the monkey typed "cbaccaabbcab..." then X would equal 3.

4.1 (*2 points*): *What's the probability $X = 3$?*

4.2 (*2 points*): *How many sequences are there of length nine that use at most two of the different keys? Hint: Be careful you don't double count!*

4.3 (*2 points*): *What's the probability $X \geq 10$?*

4.4 (*2 points*): *Prove that for any random variable Z taking values in the range $\{1, 2, 3, \ldots\}$, $E(Z) = \sum_{i=1}^{\infty} P(Z \geq i)$.*

4.5 (*2 points*): *What's the expected value of X?*

Question 5. (*15 points*)

Suppose you toss a coin four times. The sample space

$$\Omega = \{HHHH, HHHT, HHTH, \ldots, TTTT\}$$

contains 16 outcomes and you should assume each outcome is equally likely. Let X be the Binomial random variable that corresponds to the number of heads in an outcome, e.g., $X(HTHT) = 2$. Let Y be the Bernoulli random variable that evaluates to 1 if there is an even number of heads in the outcome, e.g., $Y(HHHT) = 0$ and $Y(HTHT) = 1$.

5.1 (*2 points*): *Write out the set of outcomes corresponding to $\{X = 3\}$. What is the value of $P(X = 3)$?*

5.2 (*2 points*): *Write out the set of outcomes corresponding to $\{X \geq 3\}$. What is the value of $P(X \geq 3)$?*

5.3 (*2 points*): *What is the value of $E(X)$ and $var(X)$?*

5.4 (*2 points*): *Write out the set of outcomes corresponding to $\{Y = 1\}$. What is the value of $P(Y = 1)$?*

5.5 (*2 points*): *What is the value of $E(Y)$ and $var(Y)$?*

5.6 (*2 points*): *What is the value of $P(X \geq 3 | Y = 1)$?*

5.7 (*3 points*): *Let $Z = X + Y$, e.g., $Z(HTHT) = X(HTHT) + Y(HTHT) = 2 + 1 = 3$. What are the values of*

$$P(Z = 0)\ ,\quad P(Z = 1)\ ,\quad P(Z = 3)\ ,\quad P(Z = 4)\ ,\quad P(Z = 5)\ ,\quad E(Z)\ ,\ \textit{and}\quad var(Z)\ .$$

Question 6. (*8 points*)

Consider tossing an unbiased coin until we see exactly k heads. Let Y be the random variable corresponding to the total number of coin tosses required. **Hint:** Let $Y = X_1 + X_2 + \ldots + X_k$ where X_i is the number of extra coin tosses required after the $(i-1)$th head is observed until the ith head is observed.

6.1 (*2 points*): *Compute $E(Y)$.*

6.2 (*2 points*): *Compute $var(Y)$.*

6.3 (*2 points*): *Define $Z = Y/k$. Compute $E(Z)$ and $var(Z)$.*

6.4 (*2 points*): *Use the Chebyshev bound to prove a bound on $P(|Z - E(Z)| \geq 2)$ in terms of k.*