CMPSCI 240: Reasoning Under Uncertainty
Second Midterm Exam

March 9, 2017.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- *Unless the question specifies otherwise,* you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	8+2 (Extra Credit)	
Total	58+2 (Extra Credit)	

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *The variance of a random variable can never be negative.*

1.2 (*2 points*): *For any random variable X, the events $\{X = 4\}$ and $\{X = 5\}$ are disjoint.*

1.3 (*2 points*): *If X and Y are independent binomial random variables with the same parameters, then $X + Y$ is a binomial random variable.*

1.4 (*2 points*): *Given two random variables X and Y defined on the same sample space Ω, if $X(o) < Y(o)$ for all $o \in \Omega$, then $E(X) < E(Y)$.*

1.5 (*2 points*): *For any random variable X, $var(X + Y) = 10$ if $Y = 10 - X$.*

Question 2. (*10 points*) *For this question simplify your answers fully.* Suppose a quiz consists of five TRUE/FALSE questions and suppose a student guesses the answer to each question, i.e., they are equally likely to pick either option for each question and answer each question independently.

2.1 (*2 points*): *What is the expected number of answers they get correct when they take the quiz?*

2.2 (*2 points*): *What is the variance in the number of answers they get correct?*

2.3 (*2 points*): *If the first question is worth 1 point, the second question is worth 2 points, the third question is worth 3 points etc., what is the expected number of points they receive?*

2.4 (*2 points*): *Conditioned on the fact that they got exactly four of the questions correct, what is the probability they get the first question correct?*

2.5 (*2 points*): *If they keep retaking the quiz until they get it all correct, how many times do they expect to take the quiz? You may assume they answer the questions independently each time.*

Question 3. (*10 points*) *For this question you should fully simplify your answers.* Suppose that X is a random variable where:

$$P(X = -3) = 1/3 \qquad P(X = 0) = 1/3 \qquad P(X = 3) = 1/3 \ .$$

and let $Y = X^2$.

3.1 (*2 points*): *What are the values of the following quantities:*

$$E(X) = \qquad\qquad var(X) =$$

3.2 (*2 points*): *What are the values of the following quantities:*

$$P(Y = 9) = \qquad\qquad E(Y) =$$

3.3 (*2 points*): *What are the values of the following quantities:*

$$P(X = 3|Y = 9) = \qquad\qquad P(Y = 9|X = 3) =$$

3.4 (*2 points*): *What are the values of the following quantities:*

$$E(X + Y) = \qquad\qquad E(2X + 3) =$$

For the last part of this question, suppose that Z is a random variable that takes values 1, 2, and 3. The **tail bounds** of Z are

$$P(Z \geq 1) = 1 \qquad P(Z \geq 2) = 1/2 \qquad P(Z \geq 3) = 1/4 \ .$$

3.5 (*2 points*): *What are the values of the following quantities:*

$$E(Z) = \qquad\qquad P(Z \leq 2) =$$

Question 4. (*10 points*) Suppose everyone in the class has been at UMass for exactly zero, one, or two years and that each student has taken either zero or one internships. If I were to pick a random student in the class, let Y be the number of years they have been at UMass and let I be the number of internships they have completed. The joint probabilities of Y and I are as follows:

	$I = 0$	$I = 1$
$Y = 0$	0.1	0.0
$Y = 1$	0.3	0.2
$Y = 2$	0.1	0.3

So, for example, $P(Y = 0, I = 1) = 0$.

4.1 (*3 points*): *Enter the values for the following probabilities:*

$$P(Y = 1) = \qquad P(Y = 2) = \qquad P(I = 1) =$$

4.2 (*2 points*): *Are Y and I independent? Justify your answer mathematically.*

4.3 (*4 points*): *What are the values of the following quantities?*

$$E(Y) =$$

$$E(I) =$$

$$E(YI) =$$

$$cov(Y, I) =$$

4.4 (*1 points*): *What's the probability that the student has taken the same number of internships as the number of years they've spent at UMass?*

Question 5. (*10 points*) Suppose you have a coin that lands heads with probability p and each time you toss the coin the outcome is independent. In this question your answers will be a function of p. You start tossing the coin...

5.1 (*2 points*): *Let W be the number of coin tosses until you have seen one head. E.g., if the sequence of coin tosses is* `TTHTHHH`*... then $W = 3$. What are the values of the following quantities?*

$$E(W) = \qquad\qquad P(W \geq 10) =$$

5.2 (*2 points*): *Let X be the number of coin tosses until you have seen two heads. E.g., if the sequence of coin tosses is* `TTHTHHH`*... then $X = 5$. What are the values of the following quantities?*

$$E(X) = \qquad\qquad P(X = 3) =$$

5.3 (*2 points*): *Let Y be the number of coin tosses until you have seen two heads in a row. E.g., if the sequence of coin tosses is* `TTHTHHH`*... then $Y = 6$. What are the values of the following quantities?*

$$P(Y = 20|W = 19) \quad =$$

$$P(Y = W + 1) \quad =$$

5.4 (*3 points*): *Let Z be the number of heads that occur amongst the first 100 coin tosses. What are the values of the following quantities?*

$$P(Z = 50) \quad =$$

$$P(Y \geq 101|Z = 50) \quad =$$

$$P(Y \geq 101|Z = 51) \quad =$$

5.5 (*1 points*): *In this final part of the question you may assume $p = 1/2$. Use the Chebyshev bound to prove a* lower bound *on the following quantity.*

$$P(|Z - 50| < 10) \geq$$

Question 6. (*10 points*) Suppose that the *Snack Overflow* coffee shop starts running a promotion in which every regular customer gets a limited edition coffee cup with a unique number on it. The coffee cups are numbered $1, 2, 3, \ldots$ up to either 50 or 100. Let F be the event that the highest number is 50 and let H be the event that the largest number is 100. The probability of F is 0.3 and the probability of H is 0.7. You are their first customer and are given a random cup. Let X be the number on the cup you receive.

6.1 (*3 points*): *Enter values for the following probabilities:*

$$P(X = 30|F) = \qquad P(X = 60|F) =$$

$$P(X = 30|H) = \qquad P(X = 60|H) =$$

$$P(X = 30) = \qquad P(X = 60) =$$

6.2 (*2 points*): *Enter values for the following probabilities:*

$$P(X \leq 50) = \qquad P(X \geq 51) =$$

6.3 (*3 points*): *Enter values for the following quantities:*

$$P(F|X = 30) =$$

$$P(H|X = 30) =$$

If your observe that your coffee cup has the number 30, based on the information above which is more likely: the highest number is 50 or the highest number is 100?

6.4 (*2 points*): **Extra Credit:** *Suppose instead that the cups are numbered* $1, 2, 3, \ldots, Y + 1$ *where* Y *is a Poisson random variable with parameter* $\lambda = 10$*. What is the probability that you get the coffee cup with the number 1?*

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

$$\text{For } k \in \{0,1,2,\ldots,N\} : P(X=k) = \binom{N}{k} p^k \, (1-p)^{N-k} \ , \ E(X) = Np \ , \ var(X) = Np(1-p)$$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

$$\text{For } k \in \{1,2,3,\ldots\} : \ P(X=k) = (1-p)^{k-1} \cdot p \quad , \quad E(X) = \frac{1}{p} \quad , \quad var(X) = (1-p)/p^2$$

- **Poisson Random Variable with parameter** $\lambda > 0$:

$$\text{For } k \in \{0,1,2,\ldots\} : \ P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad E(X) = \lambda \quad , \quad var(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

$$\text{For } k \in \{a, a+1, \ldots b\} : \ P(X=k) = \frac{1}{b-a+1} \ , \ E(X) = \frac{a+b}{2} \ , \ var(X) = \frac{(b-a+1)^2 - 1}{12}$$