Name: ___________________________  ID: ___________________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- Unless the question specifies otherwise, you may give your answer using arithmetic operations, such as addition, multiplication, “choose” notation and factorials (e.g., “$9 \times 35! + 2$” or “$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$” is fine).
- If you need extra space, use the back of a page.
- No books, notes, abacuses, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

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<td>$66+4$ (Extra Credit)</td>
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**Question 1.** (14 points) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (2 points): The probability of any event $A$ satisfies $0 \leq P(A) \leq 1$.

   **Answer:** TRUE.

1.2 (2 points): The expectation of a random variable can be negative.

   **Answer:** TRUE.

1.3 (2 points): For any two random variables $X$ and $Y$, then $E(X + Y) = E(X) + E(Y)$.

   **Answer:** TRUE.

1.4 (2 points): Huffman encoding never creates binary strings that all have the same length.

   **Answer:** FALSE.

1.5 (2 points): For any two events $A$ and $B$, $P(A|B) = P(B|A)$.

   **Answer:** FALSE.

1.6 (2 points): Let $X$ be the number of times I toss a fair coin until I see heads. Then $\text{var}(X) = 2$.

   **Answer:** TRUE.

1.7 (2 points): If $X$ is a Poisson random variable with parameter $\lambda$ then $P(X \geq 3\lambda) \leq \frac{1}{4\lambda}$. **Hint:** See formula sheet for expectation and variance of some common random variables.

   **Answer:** TRUE. $P(X \geq 3\lambda) \leq P(|X - E(X)| \geq 2\lambda) \leq \frac{\text{var}(X)}{4\lambda^2} = \frac{1}{4\lambda}$. 

2
**Question 2.** (10 points) An experiment has a sample space \( \Omega = \{o_1, o_2, o_3, o_4, o_5\} \) and the probability rule satisfies:

\[
P(o_1) = \frac{1}{4} \quad P(o_2) = \frac{1}{4} \quad P(o_3) = \frac{1}{4} \quad P(o_4) = \frac{1}{8} \quad P(o_5) = \frac{1}{8}
\]

Let \( A = \{o_1, o_2\} \) and \( B = \{o_2, o_3\} \). For all parts of this question, simplify your answers fully.

2.1 (3 points): What are the values of the following probabilities:

\[
P(A) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad P(A \cup B) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}
\]

2.2 (3 points): For each of the following questions, circle whether the answer is yes or no.

- Are \( A \) and \( B \) independent? **Yes** **No**
- Are \( A \) and \( B \) disjoint? **Yes** **No**
- Do \( A \) and \( B \) partition \( \Omega \)? **Yes** **No**

2.3 (4 points): Define random variables \( X \) and \( Y \) where

\[
X(o_1) = 1 \quad X(o_2) = 1 \quad X(o_3) = 2 \quad X(o_4) = 2 \quad X(o_5) = 2
\]

\[
Y(o_1) = 1 \quad Y(o_2) = 1 \quad Y(o_3) = 1 \quad Y(o_4) = 2 \quad Y(o_5) = 2
\]

What are the following values:

\[
P(X = 2) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}
\]

\[
E(X) = \frac{1}{2} \times 1 + \frac{1}{2} \times 2 = \frac{3}{2}
\]

\[
\text{var}(X) = \frac{1}{2} \times 0.5^2 + \frac{1}{2} \times 0.5^2 = \frac{1}{4}
\]

\[
E(X/Y) = \frac{3}{4} \times 1 + \frac{1}{4} \times 2 = \frac{5}{4}
\]
**Question 3.** (10 points)

3.1 (3 points): Suppose I have a coin that lands heads with probability 0.1. Suppose I start tossing the coin and let $X$ be the number of coin tosses until the first head is observed.

\[ E(X) = 10 \]
\[ P(X = 1) = 0.1 \]
\[ P(X \geq 50) = (1 - 0.1)^{49} \]

3.2 (3 points): Suppose I have a 10-sided dice that is equally likely to land on any of the 10 sides. Suppose I start throwing the dice and let $Y$ be the number of throws until it has landed on every side at least once.

\[ P(Y = 10) = \frac{10!}{10^{10}} \]
\[ E(Y) = \frac{10}{10} + \frac{10}{9} + \frac{10}{8} + \frac{10}{7} + \frac{10}{6} + \frac{10}{5} + \frac{10}{4} + \frac{10}{3} + \frac{10}{2} + \frac{10}{1} \]
\[ P(Y \leq 9) = 0 \]

3.3 (4 points): Suppose I throw 10 balls where each ball that is thrown is equally likely to land in any one of 10 bins. The bins are numbered from 1 to 10. Let $X_1$ be the number of balls that land in bin 1. Let $X_2$ be the number of balls that land in bin 2. Similarly define $X_3, \ldots, X_{10}$. Define $W = X_1 + X_2 + \ldots + X_{10}$ and $Z = \max(X_1, X_2, \ldots, X_{10})$.

\[ P(X_1 = 3) = \binom{10}{3} \times 0.1^3 \times 0.9^7 \]
\[ E(X_1 + X_2) = E(X_1) + E(X_2) = 2 \]
\[ \text{var}(W) = 0 \text{ since } W \text{ always takes the same value} \]
\[ P(Z = 1) = \frac{10!}{10^{10}} \]
Question 4. (10 points) There are two take-out restaurants in town, one called Piazza Pizza and one called Moodle Noodles. In the first two parts of this question, assume that every evening 60% of students eat at Piazza and 40% of students each at Moodle. Nobody goes to both restaurants in the same evening and each student orders exactly one meal. Furthermore, 25% of the meals sold by Piazza are vegetarian and 40% of the meals sold by Moodle are vegetarian.

4.1 (2 points): What’s the probability a randomly chosen student eats a vegetarian meal.

Answer: $0.6 \times 0.25 + 0.4 \times 0.4 = 0.31$.

4.2 (2 points): If you meet a student who has just eaten a vegetarian meal, which is more likely: they ate at Piazza or they ate at Moodle. Justify your answer.

Answer: Moodle since $0.4 \times 0.4 > 0.6 \times 0.25$.

To reduce costs both restaurants are considering only making one type of dish each evening. Suppose vegetarian meals are slightly cheaper to prepare but if both restaurants make a vegetarian meal then profits may decrease since students who are allergic to vegetables would go to a neighboring town instead. More generally, the pay-off matrix is as follows,

<table>
<thead>
<tr>
<th></th>
<th>Moodle makes Vegetarian</th>
<th>Moodle makes Non-Vegetarian</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piazza makes Vegetarian</td>
<td>4,4</td>
<td>6,5</td>
</tr>
<tr>
<td>Piazza makes Non-Vegetarian</td>
<td>5,7</td>
<td>3,3</td>
</tr>
</tbody>
</table>

E.g., if Piazza makes non-vegetarian and Moodle makes vegetarian then Piazza gets $5 profit.

4.3 (2 points): Suppose Piazza offers a vegetarian dish with probability $p$ and Moodle offers a vegetarian dish with probability $q$. What is Piazza’s expected profit?

Answer: $4pq + 6p(1-q) + 5(1-p)q + 3(1-p)(1-q) = -4pq + 3p + 2q + 3$

4.4 (2 points): If there is a Nash equilibrium with both players using mixed strategies, what is the value of $q$ in this mixed strategy?

Answer: $-4pq + 3p + 2q + 3 = p(-4q + 3) + 2q + 3$. If $q = 3/4$ then changing $p$ does not change Piazza’s expected payoff.

4.5 (2 points): What are the pure Nash equilibria?

Answer: Piazza making vegetarian and Moodle making non-vegetarian is a Nash equilibrium. Piazza making non-vegetarian and Moodle making vegetarian is a Nash equilibrium.
Question 5.  (10 points) Consider a Markov Chain with two states “1” and “0”. For \( t = 0, 1, 2, \ldots \), let \( X_t \) be the state after \( t \) steps. The initial state is 0, i.e., \( X_0 = 0 \) and the transition probabilities are
\[
P(X_i = 1|X_{i-1} = 0) = \frac{3}{4}, \quad P(X_i = 0|X_{i-1} = 0) = \frac{1}{4},
\]
\[
P(X_i = 1|X_{i-1} = 1) = \frac{1}{4}, \quad P(X_i = 0|X_{i-1} = 1) = \frac{3}{4}.
\]

5.1 (2 points): Draw the transition diagram for the Markov chain.

Answer:

5.2 (2 points): What is the value of \( P(X_2 = 1) \)?

Answer: \( P(X_2 = 1) = P(0 \rightarrow 0 \rightarrow 1) + P(0 \rightarrow 1 \rightarrow 1) = \frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} = \frac{3}{8} \).

5.3 (2 points): What is the value of \( P(X_1 = 0|X_2 = 0) \)?

Answer: \( P(X_1 = 0|X_2 = 0) = \frac{P(X_1 = 0, X_2 = 0)}{P(X_2 = 0)} = \frac{\frac{1}{4} \times \frac{1}{4}}{1 - \frac{3}{8}} = \frac{1}{10} \).

5.4 (2 points): Draw a Bayesian network for the random variables \( X_1, X_2, X_3, \) and \( X_4 \).

Answer:

5.5 (2 points): For each of the following questions, circle whether the answer is yes or no.

Are \( X_1 \) and \( X_3 \) independent?  \( \text{Yes} \quad \text{No} \)

Are events \( \{X_1 = 0\} \) and \( \{X_3 = 0\} \) independent conditioned on \( \{X_2 = 0\} \)? \( \text{Yes} \quad \text{No} \)
**Question 6.** (10 points) Recall that the distance between two binary strings is the number of positions in which they differ, e.g., \( d(101,011) = 2 \). The minimum distance of a set of binary strings is the minimum distance over all pairs of different strings in the set, e.g., the minimum distance of the set \( \{101,011,001\} \) is \( \min(d(101,011),d(101,001),d(011,001)) = 1 \).

6.1 (2 points): How many binary strings of length 6 are there that have exactly 4 one’s? What is the minimum distance of this set of strings?

\[
\text{number of binary strings} = \binom{6}{4} = 15.
\text{minimum distance} = 2.
\]

6.2 (2 points): How many binary strings of length 12 are there where the first 4 bits are identical to the middle 4 bits and the last 4 bits? What is the minimum distance of this set of strings?

\[
\text{number of binary strings} = 2^4 = 16.
\text{minimum distance} = 3.
\]

6.3 (3 points): Pick a random binary string of length 100, i.e., each bit is equally likely to be 0 or 1. Let \( X \) be the number of 1’s in this string. What are the following values?

\[
E(X) = 50 \quad \text{var}(X) = 25
\]

6.4 (3 points): Pick 10 random binary strings \( c_1,\ldots,c_{10} \) of length 100. Assume you pick each string independently.

- What’s the expected value of the distance between \( c_1 \) and \( c_2 \), i.e., \( E(d(c_1,c_2)) \)? **Answer:** 50

- Use the Chebyshev bound to give an upper bound on \( P(d(c_1,c_2) \leq 10) \). **Answer:**

\[
P(d(c_1,c_2) \leq 10) \leq P(|d(c_1,c_2) - 50| \geq 40) \leq \frac{25}{40^2} = 1/64
\]

- Use the Chebyshev bound and the union bound to give an upper bound on the probability that there is a pair of codewords whose distance is at most 10. **Answer:** \( \left(\frac{10}{2}\right) \times 1/64 = 45/64 \).
Question 7. (6 points) Amit, Brad, Coco, and Diane each put their name in a hat. They then each pick a random name (without replacement) from the hat and buy a present for the person whose name they pick. You may assume that Amit picks first, then Brad, then Coco, then Diane.

7.1 (2 points): What’s the probability nobody picks their own name?

Answer: Out of the 4! = 24 possible orderings in which the names could be drawn, the event that nobody picks their own name corresponding to 

\{BADC, BCDA, BDAC, CADB, CDAB, CDBA, DABC, DCAB, DCBA,\}

so the probability is 9/24 = 3/8.

For the rest of this question, suppose there are now 5 people. You may want to use an extension of the formula

\[ P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2). \]

Specifically,

\[
P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - \ldots - P(A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_3 \cap A_4) + \ldots + P(A_3 \cap A_4 \cap A_5)
\]

where the second line consists of all pairs of sets, the third line is all triples of sets, and the fourth line is all quadruples of sets.

7.2 (4 points): Extra Credit: Let \( A_i \) be the event that the \( i \)th person receives their own name. To get full marks you must correctly simplify your answers fully.

- What’s the value of \( P(A_1 \cap A_2 \cap A_3) \)? Answer: \( \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} = \frac{2!}{5!} = \frac{1}{60} \).

- What’s the value of \( P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_3 \cap A_4) + \ldots + P(A_3 \cap A_4 \cap A_5) \) where the sum includes all triples of sets? Answer: \( \binom{5}{3} \times \frac{2!}{5!} = \frac{1}{3!} = \frac{1}{6} \)

- What’s the probability nobody gets their own name?

Answer: \( 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = 1/2! - 1/3! + 1/4! - 1/5! = 1/2 - 1/6 + 1/24 - 1/120 = 11/30 \). Note that you can extend the line of reasoning in this question to argue that when there are \( n \) people, the probability that nobody gets their own name is

\[
\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n/n!
\]

and this tends to \( 1/e \) as \( n \) tends to infinity.
Standard Random Variables

- Bernoulli Random Variable with parameter $p \in [0, 1]$:
  \[
P(X = k) = \begin{cases} 
  1 - p & \text{if } k = 0 \\
  p & \text{if } k = 1
\end{cases}, \quad E(X) = p, \quad \text{var}(X) = p(1 - p)
\]

- Binomial Random Variable with parameters $p \in [0, 1]$ and $N \in \{1, 2, 3, \ldots\}$:
  For $k \in \{0, 1, 2, \ldots, N\}$:
  \[
P(X = k) = \binom{N}{k} p^k (1 - p)^{N-k}, \quad E(X) = Np, \quad \text{var}(X) = Np(1 - p)
\]

- Geometric Random Variable with parameter $p \in [0, 1]$:
  For $k \in \{1, 2, 3, \ldots\}$:
  \[
P(X = k) = (1 - p)^{k-1} \cdot p, \quad E(X) = \frac{1}{p}, \quad \text{var}(X) = (1 - p)/p^2
\]

- Poisson Random Variable with parameter $\lambda > 0$:
  For $k \in \{0, 1, 2, \ldots\}$:
  \[
P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad E(X) = \lambda, \quad \text{var}(X) = \lambda
\]

- Discrete Uniform Random Variable with parameters $a, b \in \mathbb{Z}$ and $a < b$:
  For $k \in \{a, a+1, \ldots b\}$:
  \[
P(X = k) = \frac{1}{b - a + 1}, \quad E(X) = \frac{a + b}{2}, \quad \text{var}(X) = \frac{(b - a + 1)^2 - 1}{12}
\]

Bayes Formula

- If $A_1, \ldots, A_n$ partition $\Omega$ then for any event $B$:
  \[
P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}
\]