Instructions:

• Answer the questions directly on the exam pages.

• Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.

• Unless the question specifies otherwise, you may give your answer using arithmetic operations, such as addition, multiplication, “choose” notation and factorials (e.g., “9 × 35! + 2” or “0.5 × 0.3/(0.2 × 0.5 + 0.9 × 0.1)” is fine).

• If you need extra space, use the back of a page.

• No books, notes, abacuses, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.

• If you have questions during the exam, raise your hand.

• The formulas for some standard random variables can be found on the last page.

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<td>5</td>
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<td>Total</td>
<td>66+4 (Extra Credit)</td>
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**Question 1.** (14 points) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (2 points): The probability of any event $A$ satisfies $0 \leq P(A) \leq 1$.

1.2 (2 points): The expectation of a random variable can be negative.

1.3 (2 points): For any two random variables $X$ and $Y$, then $E(X + Y) = E(X) + E(Y)$.

1.4 (2 points): Huffman encoding never creates binary strings that all have the same length.

1.5 (2 points): For any two events $A$ and $B$, $P(A|B) = P(B|A)$.

1.6 (2 points): Let $X$ be the number of times I toss a fair coin until I see heads. Then $\text{var}(X) = 2$.

1.7 (2 points): If $X$ is a Poisson random variable with parameter $\lambda$ then $P(X \geq 3\lambda) \leq \frac{1}{4\lambda}$. **Hint:** See formula sheet for expectation and variance of some common random variables.
Question 2. (10 points) An experiment has a sample space $\Omega = \{o_1, o_2, o_3, o_4, o_5\}$ and the probability rule satisfies:

\begin{align*}
P(o_1) &= 1/4 \\
P(o_2) &= 1/4 \\
P(o_3) &= 1/4 \\
P(o_4) &= 1/8 \\
P(o_5) &= 1/8
\end{align*}

Let $A = \{o_1, o_2\}$ and $B = \{o_2, o_3\}$. For all parts of this question, simplify your answers fully.

2.1 (3 points): What are the values of the following probabilities:

\begin{align*}
P(A) = \\
P(A \cup B) = \\
P(A|B) =
\end{align*}

2.2 (3 points): For each of the following questions, circle whether the answer is yes or no.

\begin{itemize}
  \item Are $A$ and $B$ independent? \hspace{1cm} Yes \hspace{1cm} No
  \item Are $A$ and $B$ disjoint? \hspace{1cm} Yes \hspace{1cm} No
  \item Do $A$ and $B$ partition $\Omega$? \hspace{1cm} Yes \hspace{1cm} No
\end{itemize}

2.3 (4 points): Define random variables $X$ and $Y$ where

\begin{align*}
X(o_1) &= 1 & X(o_2) &= 1 & X(o_3) &= 2 & X(o_4) &= 2 & X(o_5) &= 2 \\
Y(o_1) &= 1 & Y(o_2) &= 1 & Y(o_3) &= 1 & Y(o_4) &= 2 & Y(o_5) &= 2
\end{align*}

What are the following values:

\begin{align*}
P(X = 2) = \\
E(X) = \\
\text{var}(X) = \\
E(X/Y) =
\end{align*}
Question 3. (10 points)

3.1 (3 points): Suppose I have a coin that lands heads with probability 0.1. Suppose I start tossing the coin and let \( X \) be the number of coin tosses until the first head is observed.

\[
E(X) = \\
P(X = 1) = \\
P(X \geq 50) = 
\]

3.2 (3 points): Suppose I have a 10-sided dice that is equally likely to land on any of the 10 sides. Suppose I start throwing the dice and let \( Y \) be the number of throws until it has landed on every side at least once.

\[
P(Y = 10) = \\
E(Y) = \\
P(Y \leq 9) = 
\]

3.3 (4 points): Suppose I throw 10 balls where each ball that is thrown is equally likely to land in any one of 10 bins. The bins are numbered from 1 to 10. Let \( X_1 \) be the number of balls that land in bin 1. Let \( X_2 \) be the number of balls that land in bin 2. Similarly define \( X_3, \ldots, X_{10} \). Define \( W = X_1 + X_2 + \ldots + X_{10} \) and \( Z = \max(X_1, X_2, \ldots, X_{10}) \).

\[
P(X_1 = 3) = \\
E(X_1 + X_2) = \\
\text{var}(W) = \\
P(Z = 1) = 
\]
Question 4. (10 points) There are two take-out restaurants in town, one called Piazza Pizza and one called Moodle Noodles. In the first two parts of this question, assume that every evening 60% of students eat at Piazza and 40% of students each at Moodle. Nobody goes to both restaurants in the same evening and each student orders exactly one meal. Furthermore, 25% of the meals sold by Piazza are vegetarian and 40% of the meals sold by Moodle are vegetarian.

4.1 (2 points): What’s the probability a randomly chosen student eats a vegetarian meal.

4.2 (2 points): If you meet a student who has just eaten a vegetarian meal, which is more likely: they ate at Piazza or they ate at Moodle. Justify your answer.

To reduce costs both restaurants are considering only making one type of dish each evening. Suppose vegetarian meals are slightly cheaper to prepare but if both restaurants make a vegetarian meal then profits may decrease since students who are allergic to vegetables would go to a neighboring town instead. More generally, the pay-off matrix is as follows,

<table>
<thead>
<tr>
<th></th>
<th>Moodle makes Vegetarian</th>
<th>Moodle makes Non-Vegetarian</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piazza makes Vegetarian</td>
<td>4,4</td>
<td>6,5</td>
</tr>
<tr>
<td>Piazza makes Non-Vegetarian</td>
<td>5,7</td>
<td>3,3</td>
</tr>
</tbody>
</table>

E.g., if Piazza makes non-vegetarian and Moodle makes vegetarian then Piazza gets $5 profit.

4.3 (2 points): Suppose Piazza offers a vegetarian dish with probability $p$ and Moodle offers a vegetarian dish with probability $q$. What is Piazza’s expected profit?

4.4 (2 points): If there is a Nash equilibrium with both players using mixed strategies, what is the value of $q$ in this mixed strategy?

4.5 (2 points): What are the pure Nash equilibria?
**Question 5.** (10 points) Consider a Markov Chain with two states “1” and “0”. For \( t = 0, 1, 2, \ldots \), let \( X_t \) be the state after \( t \) steps. The initial state is 0, i.e., \( X_0 = 0 \) and the transition probabilities are

\[
\begin{align*}
P(X_i = 1|X_{i-1} = 0) &= 3/4 , & P(X_i = 0|X_{i-1} = 0) &= 1/4 , \\
P(X_i = 1|X_{i-1} = 1) &= 1/4 , & P(X_i = 0|X_{i-1} = 1) &= 3/4 .
\end{align*}
\]

5.1 (2 points): Draw the transition diagram for the Markov chain.

5.2 (2 points): What is the value of \( P(X_2 = 1) \)?

5.3 (2 points): What is the value of \( P(X_1 = 0|X_2 = 0) \)?

5.4 (2 points): Draw a Bayesian network for the random variables \( X_1, X_2, X_3, \) and \( X_4 \).

5.5 (2 points): For each of the following questions, circle whether the answer is yes or no.

Are \( X_1 \) and \( X_3 \) independent? \hspace{1cm} Yes \hspace{1cm} No

Are events \( \{X_1 = 0\} \) and \( \{X_3 = 0\} \) independent conditioned on \( \{X_2 = 0\} \)? \hspace{1cm} Yes \hspace{1cm} No
**Question 6.** (10 points) Recall that the distance between two binary strings is the number of positions in which they differ, e.g., \(d(101, 011) = 2\). The minimum distance of a set of binary strings is the minimum distance over all pairs of different strings in the set, e.g., the minimum distance of the set \(\{101, 011, 001\}\) is \(\min(d(101, 011), d(101, 001), d(011, 001)) = 1\).

**6.1** (2 points): How many binary strings of length 6 are there that have exactly 4 one’s? What is the minimum distance of this set of strings?

\[
\begin{align*}
\text{number of binary strings} &= \\
\text{minimum distance} &=
\end{align*}
\]

**6.2** (2 points): How many binary strings of length 12 are there where the first 4 bits are identical to the middle 4 bits and the last 4 bits? What is the minimum distance of this set of strings?

\[
\begin{align*}
\text{number of binary strings} &= \\
\text{minimum distance} &=
\end{align*}
\]

**6.3** (3 points): Pick a random binary string of length 100, i.e., each bit is equally likely to be 0 or 1. Let \(X\) be the number of 1’s in this string. What are the following values?

\[
\begin{align*}
E(X) &= \\
\operatorname{var}(X) &=
\end{align*}
\]

**6.4** (3 points): Pick 10 random binary strings \(c_1, \ldots, c_{10}\) of length 100. Assume you pick each string independently.

- What’s the expected value of the distance between \(c_1\) and \(c_2\), i.e., \(E(d(c_1, c_2))\)?

- Use the Chebyshev bound to give an upper bound on \(P(d(c_1, c_2) \leq 10)\).

- Use the Chebyshev bound and the union bound to give an upper bound on the probability that there is a pair of codewords whose distance is at most 10.
Question 7. (6 points) Amit, Brad, Coco, and Diane each put their name in a hat. They then each pick a random name (without replacement) from the hat and buy a present for the person whose name they pick. You may assume that Amit picks first, then Brad, then Coco, then Diane.

7.1 (2 points): What’s the probability nobody picks their own name?

For the rest of this question, suppose there are now 5 people. You may want to use an extension of the formula \( P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \). Specifically,

\[
P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)
- P(A_1 \cap A_2) - P(A_1 \cap A_3) - \ldots - P(A_4 \cap A_5)
+ P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_3 \cap A_4) + \ldots + P(A_3 \cap A_4 \cap A_5)
- P(A_1 \cap A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_3 \cap A_4 \cap A_5) - \ldots - P(A_2 \cap A_3 \cap A_4 \cap A_5)
+ P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)
\]

where the second line consists of all pairs of sets, the third line is all triples of sets, and the fourth line is all quadruples of sets.

7.2 (4 points): Extra Credit: Let \( A_i \) be the event that the \( i \)th person receives their own name. To get full marks you must correctly simplify your answers fully.

- What’s the value of \( P(A_1 \cap A_2 \cap A_3) \)?

- What’s the value of \( P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_3 \cap A_4) + \ldots + P(A_3 \cap A_4 \cap A_5) \) where the sum includes all triples of sets?

- What’s the probability nobody gets their own name?
Standard Random Variables

- Bernoulli Random Variable with parameter $p \in [0,1]$:
  \[
P(X = k) = \begin{cases} 
  1 - p & \text{if } k = 0 \\
  p & \text{if } k = 1 
\end{cases}, \quad E(X) = p, \quad var(X) = p(1-p)
\]

- Binomial Random Variable with parameters $p \in [0,1]$ and $N \in \{1,2,3,\ldots\}$:
  For $k \in \{0,1,2,\ldots,N\}$: $P(X = k) = \binom{N}{k} p^k (1-p)^{N-k}$, $E(X) = Np$, $var(X) = Np(1-p)$

- Geometric Random Variable with parameter $p \in [0,1]$:
  For $k \in \{1,2,3,\ldots\}$: $P(X = k) = (1-p)^{k-1} \cdot p$, $E(X) = \frac{1}{p}$, $var(X) = (1-p)/p^2$

- Poisson Random Variable with parameter $\lambda > 0$:
  For $k \in \{0,1,2,\ldots\}$: $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $E(X) = \lambda$, $var(X) = \lambda$

- Discrete Uniform Random Variable with parameters $a,b \in \mathbb{Z}$ and $a < b$:
  For $k \in \{a,a+1,\ldots,b\}$: $P(X = k) = \frac{1}{b-a+1}$, $E(X) = \frac{a+b}{2}$, $var(X) = \frac{(b-a+1)^2 - 1}{12}$

Bayes Formula

- If $A_1,\ldots,A_n$ partition $\Omega$ then for any event $B$:
  \[
P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}
\]