CMPSCI 240: Reasoning Under Uncertainty
Final Exam

May 9, 2017.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- *Unless the question specifies otherwise,* you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, abacuses, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	14	
2	10	
3	10	
4	10	
5	10	
6	10	
7	2+4 (Extra Credit)	
Total	66+4 (Extra Credit)	

Question 1. (*14 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *The probability of any event A satisfies $0 \leq P(A) \leq 1$.*

1.2 (*2 points*): *The expectation of a random variable can be negative.*

1.3 (*2 points*): *For any two random variables X and Y, then $E(X + Y) = E(X) + E(Y)$.*

1.4 (*2 points*): *Huffman encoding never creates binary strings that all have the same length.*

1.5 (*2 points*): *For any two events A and B, $P(A|B) = P(B|A)$.*

1.6 (*2 points*): *Let X be the number of times I toss a fair coin until I see heads. Then $var(X) = 2$.*

1.7 (*2 points*): *If X is a Poisson random variable with parameter λ then $P(X \geq 3\lambda) \leq \frac{1}{4\lambda}$.* **Hint:** *See formula sheet for expectation and variance of some common random variables.*

Question 2. (*10 points*) An experiment has a sample space $\Omega = \{o_1, o_2, o_3, o_4, o_5\}$ and the probability rule satisfies:

$$P(o_1) = 1/4 \quad P(o_2) = 1/4 \quad P(o_3) = 1/4 \quad P(o_4) = 1/8 \quad P(o_5) = 1/8$$

Let $A = \{o_1, o_2\}$ and $B = \{o_2, o_3\}$. For all parts of this question, simplify your answers fully.

2.1 (*3 points*): *What are the values of the following probabilities:*

$P(A) =$ $P(A \cup B) =$ $P(A|B) =$

2.2 (*3 points*): *For each of the following questions, circle whether the answer is yes or no.*

Are A and B independent?	***Yes***	***No***
Are A and B disjoint?	***Yes***	***No***
Do A and B partition Ω?	***Yes***	***No***

2.3 (*4 points*): *Define random variables X and Y where*

$$X(o_1) = 1 \quad X(o_2) = 1 \quad X(o_3) = 2 \quad X(o_4) = 2 \quad X(o_5) = 2$$

$$Y(o_1) = 1 \quad Y(o_2) = 1 \quad Y(o_3) = 1 \quad Y(o_4) = 2 \quad Y(o_5) = 2$$

What are the following values:

$$P(X = 2) \;=$$

$$E(X) \;=$$

$$var(X) \;=$$

$$E(X/Y) \;=$$

Question 3. (*10 points*)

3.1 (*3 points*): *Suppose I have a coin that lands heads with probability 0.1. Suppose I start tossing the coin and let X be the number of coin tosses until the first head is observed.*

$$E(X) =$$

$$P(X = 1) =$$

$$P(X \geq 50) =$$

3.2 (*3 points*): *Suppose I have a 10-sided dice that is equally likely to land on any of the 10 sides. Suppose I start throwing the dice and let Y be the number of throws until it has landed on every side at least once.*

$$P(Y = 10) =$$

$$E(Y) =$$

$$P(Y \leq 9) =$$

3.3 (*4 points*): *Suppose I throw 10 balls where each ball that is thrown is equally likely to land in any one of 10 bins. The bins are numbered from 1 to 10. Let X_1 be the number of balls that land in bin 1. Let X_2 be the number of balls that land in bin 2. Similarly define $X_3, \ldots, X_{10}$. Define $W = X_1 + X_2 + \ldots + X_{10}$ and $Z = \max(X_1, X_2, \ldots, X_{10})$.*

$$P(X_1 = 3) =$$

$$E(X_1 + X_2) =$$

$$var(W) =$$

$$P(Z = 1) =$$

Question 4. (*10 points*) There are two take-out restaurants in town, one called Piazza Pizza and one called Moodle Noodles. In the first two parts of this question, assume that every evening 60% of students eat at Piazza and 40% of students each at Moodle. Nobody goes to both restaurants in the same evening and each student orders exactly one meal. Furthermore, 25% of the meals sold by Piazza are vegetarian and 40% of the meals sold by Moodle are vegetarian.

4.1 (*2 points*): *What's the probability a randomly chosen student eats a vegetarian meal.*

4.2 (*2 points*): *If you meet a student who has just eaten a vegetarian meal, which is more likely: they ate at Piazza or they ate at Moodle. Justify your answer.*

To reduce costs both restaurants are considering only making *one* type of dish each evening. Suppose vegetarian meals are slightly cheaper to prepare but if both restaurants make a vegetarian meal then profits may decrease since students who are allergic to vegetables would go to a neighboring town instead. More generally, the pay-off matrix is as follows,

	Moodle makes Vegetarian	Moodle makes Non-Vegetarian
Piazza makes Vegetarian	4,4	6,5
Piazza makes Non-Vegetarian	5,7	3,3

E.g., if Piazza makes non-vegetarian and Moodle makes vegetarian then Piazza gets \$5 profit.

4.3 (*2 points*): *Suppose Piazza offers a vegetarian dish with probability p and Moodle offers a vegetarian dish with probability q. What is Piazza's expected profit?*

4.4 (*2 points*): *If there is a Nash equilibrium with both players using mixed strategies, what is the value of q in this mixed strategy?*

4.5 (*2 points*): *What are the pure Nash equilibria?*

Question 5. (*10 points*) Consider a Markov Chain with two states "1" and "0". For $t = 0, 1, 2, \ldots$, let X_t be the state after t steps. The initial state is 0, i.e., $X_0 = 0$ and the transition probabilities are

$$P(X_i = 1|X_{i-1} = 0) = 3/4 \ , \quad P(X_i = 0|X_{i-1} = 0) = 1/4 \ ,$$

$$P(X_i = 1|X_{i-1} = 1) = 1/4 \ , \quad P(X_i = 0|X_{i-1} = 1) = 3/4 \ .$$

5.1 (*2 points*): *Draw the transition diagram for the Markov chain.*

5.2 (*2 points*): *What is the value of* $P(X_2 = 1)$*?*

5.3 (*2 points*): *What is the value of* $P(X_1 = 0|X_2 = 0)$*?*

5.4 (*2 points*): *Draw a Bayesian network for the random variables* X_1, X_2, X_3*, and* X_4*.*

5.5 (*2 points*): *For each of the following questions, circle whether the answer is yes or no.*

Are X_1 *and* X_3 *independent?* ***Yes*** ***No***

Are events $\{X_1 = 0\}$ *and* $\{X_3 = 0\}$ *independent conditioned on* $\{X_2 = 0\}$*?* ***Yes*** ***No***

Question 6. (*10 points*) Recall that the distance between two binary strings is the number of positions in which they differ, e.g., $d(\texttt{101}, \texttt{011}) = 2$. The minimum distance of a set of binary strings is the minimum distance over all pairs of different strings in the set, e.g., the minimum distance of the set $\{\texttt{101}, \texttt{011}, \texttt{001}\}$ is $\min(d(\texttt{101}, \texttt{011}), d(\texttt{101}, \texttt{001}), d(\texttt{011}, \texttt{001})) = 1$.

6.1 (*2 points*): *How many binary strings of length 6 are there that have exactly 4 one's? What is the minimum distance of this set of strings?*

number of binary strings = minimum distance =

6.2 (*2 points*): *How many binary strings of length 12 are there where the first 4 bits are identical to the middle 4 bits and the last 4 bits? What is the minimum distance of this set of strings?*

number of binary strings = minimum distance =

6.3 (*3 points*): *Pick a random binary string of length 100, i.e., each bit is equally likely to be 0 or 1. Let X be the number of 1's in this string. What are the following values?*

$E(X) =$ $var(X) =$

6.4 (*3 points*): *Pick 10 random binary strings $c_1, \ldots, c_{10}$ of length 100. Assume you pick each string independently.*

- *What's the expected value of the distance between c_1 and c_2, i.e., $E(d(c_1, c_2))$?*

- *Use the Chebyshev bound to give an upper bound on $P(d(c_1, c_2) \leq 10)$.*

- *Use the Chebyshev bound and the union bound to give an upper bound on the probability that there is a pair of codewords whose distance is at most* 10.

Question 7. (*6 points*) Amit, Brad, Coco, and Diane each put their name in a hat. They then each pick a random name (without replacement) from the hat and buy a present for the person whose name they pick. You may assume that Amit picks first, then Brad, then Coco, then Diane.

7.1 (*2 points*): *What's the probability nobody picks their own name?*

For the rest of this question, suppose there are now 5 people. You may want to use an extension of the formula $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. Specifically,

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = & P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) \\ & - P(A_1 \cap A_2) - P(A_1 \cap A_3) - \ldots - P(A_4 \cap A_5) \\ & + P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_3 \cap A_4) + \ldots + P(A_3 \cap A_4 \cap A_5) \\ & - P(A_1 \cap A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_3 \cap A_4 \cap A_5) - \ldots - P(A_2 \cap A_3 \cap A_4 \cap A_5) \\ & + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \end{aligned}$$

where the second line consists of all pairs of sets, the third line is all triples of sets, and the fourth line is all quadruples of sets.

7.2 (*4 points*): **Extra Credit:** *Let A_i be the event that the ith person receives their own name. To get full marks you must correctly simplify your answers fully.*

- *What's the value of $P(A_1 \cap A_2 \cap A_3)$?*

- *What's the value of $P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_3 \cap A_4) + \ldots + P(A_3 \cap A_4 \cap A_5)$ where the sum includes all triples of sets?*

- *What's the probability nobody gets their own name?*

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

$$\text{For } k \in \{0,1,2,\ldots,N\} : P(X=k) = \binom{N}{k} p^k (1-p)^{N-k} \ , \ E(X) = Np \ , \ var(X) = Np(1-p)$$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

$$\text{For } k \in \{1,2,3,\ldots\} : \ P(X=k) = (1-p)^{k-1} \cdot p \quad , \quad E(X) = \frac{1}{p} \quad , \quad var(X) = (1-p)/p^2$$

- **Poisson Random Variable with parameter** $\lambda > 0$:

$$\text{For } k \in \{0,1,2,\ldots\} : \ P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad E(X) = \lambda \quad , \quad var(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

$$\text{For } k \in \{a, a+1, \ldots b\} : \ P(X=k) = \frac{1}{b-a+1} \ , \ E(X) = \frac{a+b}{2} \ , \ var(X) = \frac{(b-a+1)^2 - 1}{12}$$

Bayes Formula

- If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$