CMPSCI 240: Reasoning Under Uncertainty
Second Midterm Exam SOLUTIONS

March 9, 2016.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- *Unless the question specifies otherwise,* you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	9+1 (Extra Credit)	
Total	59+1 (Extra Credit)	

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *For every random variable that takes integer values, the expectation is an integer.*

Answer: FALSE.

1.2 (*2 points*): *For any random variable X, the events $\{X = 1\}$ and $\{X = 2\}$ are independent.*

Answer: FALSE.

1.3 (*2 points*): $P(|X - 1| \geq 2) = P((X - 1)^2 \geq 4)$ *for any random variable X.*

Answer: TRUE.

1.4 (*2 points*): *Two random variables with the same variance must have the same expectation.*

Answer: FALSE.

1.5 (*2 points*): $\sum_{k=1}^{N} \binom{N}{k} p^k (1-p)^{N-k} = 1 - (1-p)^N$ *for any $p \in [0, 1]$ and $N \in \{1, 2, 3, \ldots\}$.*

Answer: TRUE.

Question 2. (*10 points*) Suppose that X is a random variable where:

$$P(X = -4) = 1/4 \quad P(X = 0) = 1/4 \quad P(X = 2) = 1/2 \; .$$

and let $Y = (X + 4)(X - 2)$.

2.1 (*2 points*): *What are the values of the following quantities:*

$E(X) = 1/4 \times (-4) + 1/4 \times 0 + 1/2 \times 2 = 0$ $var(X) = 1/4 \times (-4 - 0)^2 + 1/4 \times (0 - 0)^2 + 1/2 \times (2 - 0)^2 = 6$

2.2 (*2 points*): *What are the values of the following quantities:*

$P(Y = -8) = P(X = 0) = 1/4$ $P(Y = 0) = P(X = -4) + P(X = 2) = 3/4$

2.3 (*2 points*): *What are the values of the following quantities:*

$E(Y) = (-8) \times 1/4 + 0 \times 3/4 = -2$ $var(Y) = (-8 + 2)^2 \times 1/4 + (0 + 2)^2 \times 3/4 = 12$

2.4 (*3 points*): *What are the values of the following quantities:*

$$E(XY) = 0 \qquad E(X)E(Y) = 0$$

Are X and Y independent? *No.*

2.5 (*1 points*): *What is the value of the following quantity:*

$$P(X = 0|Y = 0) = 0 \ .$$

Question 3. (*10 points*)

Suppose a student takes a Moodle quiz containing two questions. Let Q be the number of questions she gets correct. Let S be the number of hours she had studied that week. Suppose S takes the value 0, 1, or 2. If we pick a random student from the class, the joint probabilities of Q and S are as follows:

	$S = 0$	$S = 1$	$S = 2$
$Q = 0$	0.1	0.1	0
$Q = 1$	0.1	0.2	0
$Q = 2$	0	0	0.5

So, for example, $P(Q = 1, S = 1) = 0.2$.

3.1 (*2 points*): *Enter the values for the following probabilities:*

$$P(Q = 1) = 0.1 + 0.2 + 0 = 0.3 \qquad P(Q = 2) = 0 + 0 + 0.5 = 0.5$$

$$P(S = 1) = 0.1 + 0.2 + 0 = 0.3 \qquad P(S = 2) = 0 + 0 + 0.5 = 0.5$$

3.2 (*2 points*): *Are Q and S independent? Justify your answer.*

Answer: No. E.g., $P(Q = 2, S = 2) = 0.5 \neq 0.5^2 = P(Q = 2)P(S = 2)$.

3.3 (*4 points*): *What are the values of the following quantities?*

$$E(Q) = 0.3 \times 1 + 0.5 \times 2 = 1.3$$

$$E(S) = 0.3 \times 1 + 0.5 \times 2 = 1.3$$

$$E(QS) = 0.2 \times 1 + 0.5 \times 4 = 2.2$$

$$cov(Q, S) = 2.2 - 1.3^2 = 0.51$$

3.4 (*2 points*): *Suppose the instructor gives 2 points for every question the students gets right and 1 point for every hour she studies. Let P be the number points a student receives. Write a formula for P in terms of Q and S and compute the expectation of P.*

$$P = 2Q + S$$

$$E(P) = 2E(Q) + E(S) = 3.9$$

Question 4. (*10 points*) You have two coins in your pocket: one coin is fair and the other coin lands heads with probability 3/4. You pick one of the coins at random, i.e., you pick the fair coin with probability 1/2 and the biased coin with probability 1/2. Let F be the event that you pick the fair coin and let B be the event you pick the biased coin. You then toss the coin four times and let X be the number of heads that are observed.

4.1 (*4 points*): *Enter values for the following probabilities:*

$$P(F) = 1/2 \qquad P(B) = 1/2$$

$$P(X = 2|F) = \binom{4}{2} \times 1/2^4 = 3/8 \qquad P(X = 0|B) = 1/4^4 = 1/256$$

4.2 (*2 points*): *Enter values for the following probabilities:*

$$P(\{X = 2\} \cap F) = P(F)P(X = 2|F) = 1/2 \times \binom{4}{2} \times 1/2^4 = 3/16$$

$$P(X \geq 3|B) = P(X = 3|B) + P(X = 4|B) = \binom{4}{3}(3/4)^3 \times 1/4 + (3/4)^4 = 189/256$$

4.3 (*2 points*): *Enter values for the following quantities:*

$$P(X = 2) = P(\{X = 2\} \cap F) + P(\{X = 2\} \cap B) = 1/2\left(\binom{4}{2}(1/2)^4 + \binom{4}{2}(3/4)^2(1/4)^2\right) = 75/256$$

$$E(X) = 1/2(2 + 3) = 2.5$$

4.4 (*2 points*): *Conditioned on the event $\{X = 4\}$, what is the probability of F?*

Answer:

$$P(F|X = 4) = \frac{P(\{X = 4\} \cap F)}{P(X = 4)} = \frac{1/2 \times \binom{4}{2}(1/2)^4}{1/2\left(\binom{4}{2}(1/2)^4 + \binom{4}{2}(3/4)^4\right)} = \frac{(1/2)^4}{(1/2)^4 + (3/4)^4} = 16/97$$

Question 5. (*10 points*) Suppose X is the number of dust storms that occur on Mars next year. You should assume that X is a discrete uniform random variable that take one of the 101 values in the range $\{0, 1, 2, 3, \ldots, 100\}$. Let $Y = |X - E(X)|$. **Hint:** Remember there's a formula sheet at the back of the exam.

5.1 (*3 points*): *Enter values for the following probabilities:*

$$E(X) = (100 - 0)/2 = 50$$

$$var(X) = ((100 - 0 + 1)^2 - 1)/12 = 850$$

$$P(X = 12) = 1/101$$

5.2 (*2 points*): *Enter the following values. You may use the fact* $1 + 2 + \ldots + 50 = 1275$.

$$P(Y = 0) = 1/101$$

$$P(Y = 1) = 2/101$$

$$P(Y = 2) = 2/101$$

$$E(Y) = (1 + 2 + \ldots + 50) \times 2/101 = 2550/101$$

5.3 (*2 points*): *By applying the Markov Bound to* Y*, give an upper bound for the following quantity:*

$$P(|X - E(X)| \geq 30) = P(Y \geq 30) \leq E(Y)/30 = 2550/3030 = 85/101 = 0.84\ldots$$

5.4 (*2 points*): *By applying the Chebyshev Bound, give an upper bound for the following quantity:*

$$P(|X - E(X)| \geq 30) \leq var(X)/30^2 = 850/900 = 17/18 = 0.94\ldots$$

5.5 (*1 points*): *What is the exact value of* $P(|X - E(X)| \geq 30)$*?*

Answer:

$$P(|X - E(X)| \geq 30) = P(Y = 30) + P(Y = 31) + \ldots + P(Y = 50) = 21 \times 2/101 = 42/101 = 0.41\ldots$$

Question 6. (*10 points*) Suppose that a course has an optional discussion section. There are 99 students assigned to this discussion and, conveniently, their student IDs are $1, 2, \ldots, 99$. Each student goes to the discussion section with probability $2/3$ and the students make this decision independently of each other. Let $X_i = 1$ if the student with ID i turns up and $X_i = 0$ if they don't. Let $X = X_1 + X_2 + \ldots + X_{99}$ be the total number of students that go to the discussion section.

6.1 (*3 points*): *What are the values of the following quantities:*

$$E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 3 \times 2/3 = 2$$

$$var(X_1 + X_2 + X_3) = var(X_1) + var(X_2) + var(X_3) = 3 \times 2/3 \times 1/3 = 2/3$$

$$P(X_1 + X_2 + X_3 = 2) = \binom{3}{2}(2/3)^2 \times 1/3 = 4/9$$

6.2 (*2 points*): *Are the random variables X and X_1 independent? Justify your answer.*

Answer: No. For example, $P(X = 99, X_1 = 0) = 0 \neq P(X = 99)P(X_1 = 0)$.

6.3 (*2 points*): *What is the value of $P(X_2 = 1|X = 66)$? Are the events $\{X = 66\}$ and $\{X_2 = 1\}$ independent? Justify your answer.*

Answer:

$$P(X_2 = 1|X = 66) = \frac{P(X_2 = 1, X = 66)}{P(X = 66)} = \frac{2/3 \times \binom{98}{65} \times (2/3)^{65} \times (1/3)^{33}}{\binom{99}{66} \times (2/3)^{66} \times (1/3)^{33}} = \frac{\binom{98}{65}}{\binom{99}{66}} = \frac{66}{99} = \frac{2}{3}.$$

Since this equals $P(X_2 = 1) = 2/3$, the events are independent.

6.4 (*2 points*): *The instructor decides to start bringing cake to the discussion sections. If $k > 0$ students attend the discussion, each gets a $1/k$ fraction of the cake. Let Y_i be the fraction of cake that student i receives. Note that $Y_i = 0$ if $X_i = 0$. $X = 0$ implies nobody gets any cake. What are the following values:*

$$E(Y_1 + Y_2 + Y_3 + \ldots + Y_{99}) = P(X \neq 0) = 1 - (1/3)^{99}.$$

$$P(Y_1 = 1) = 2/3 \times (1/3)^{98}$$

6.5 (*1 points*): **Extra Credit:** *What is the value of $E(Y_1)$? Simplify your answer fully.*

Answer: Since $E(Y_1) + E(Y_2) + \ldots + E(Y_{99}) = E(Y_1 + Y_2 + Y_3 + \ldots + Y_{99}) = 1 - (1/3)^{99}$ and $E(Y_1) = E(Y_2) = \ldots = E(Y_{99})$, it follows that $E(Y_1) = \frac{1-(1/3)^{99}}{99}$.

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

$$\text{For } k \in \{0,1,2,\ldots,N\} : P(X=k) = \binom{N}{k} p^k (1-p)^{N-k} \ , \ E(X) = Np \ , \ var(X) = Np(1-p)$$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

$$\text{For } k \in \{1,2,3,\ldots\} : \ P(X=k) = (1-p)^{k-1} \cdot p \quad , \quad E(X) = \frac{1}{p} \quad , \quad var(X) = (1-p)/p^2$$

- **Poisson Random Variable with parameter** $\lambda > 0$:

$$\text{For } k \in \{0,1,2,\ldots\} : \ P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad E(X) = \lambda \quad , \quad var(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

$$\text{For } k \in \{a, a+1, \ldots b\} : \ P(X=k) = \frac{1}{b-a+1} \ , \ E(X) = \frac{a+b}{2} \ , \ var(X) = \frac{(b-a+1)^2 - 1}{12}$$