CMPSCI 240: Reasoning Under Uncertainty
First Midterm Exam

February 17, 2016.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, *unless the problem says otherwise*, you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	8+2 (Extra Credit)	
5	10	
6	10	
Total	58+2 (Extra Credit)	

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *For any event* A, $P(A \cup A^c) = 1$.

Answer: TRUE.

1.2 (*2 points*): *If* A *and* B *are any two disjoint events then* $P(A \cap B) = P(A)P(B)$.

Answer: FALSE.

1.3 (*2 points*): $10^{10} \geq 10!$.

Answer: TRUE.

1.4 (*2 points*): *For any two events* A *and* B *where* $A \subseteq B$ *then* $P(A) \leq P(B)$.

Answer: TRUE.

1.5 (*2 points*): $\binom{20}{8} = \binom{20}{12}$.

Answer: TRUE. Note that $\binom{20}{8} = \frac{20!}{8! \times 12!} = \frac{20!}{12! \times 8!} = \binom{20}{12}$.

Question 2. (*10 points*) Suppose you perform an experiment where the sample space is

$$\Omega = \{o_1, o_2, o_3, o_4, o_5\}$$

and the probability rule satisfies:

$$P(\{o_1\}) = 1/2 \quad , \quad P(\{o_2\}) = 1/3 \quad , \quad P(\{o_3\}) = 1/18 \quad , \quad P(\{o_4\}) = 1/18 \quad , \quad P(\{o_5\}) = 1/18$$

Define the events $A = \{o_1, o_2\}$, $B = \{o_2, o_3\}$, and $C = \{o_4, o_5\}$.

2.1 (*2 points*): *What is the value of* $P(A)$*?*

Answer: $P(A) = P(\{o_1\}) + P(\{o_2\}) = 1/2 + 1/3 = 5/6$.

2.2 (*2 points*): *What is the value of* $P(A \cup B)$*?*

Answer: $P(A \cup B) = P(\{o_1\}) + P(\{o_2\}) + P(\{o_3\}) = 1/2 + 1/3 + 1/18 = 8/9.$

2.3 (*2 points*): *What is the value of* $P(A \cap B)$*?*

Answer: $P(A \cap B) = P(\{o_2\}) = 1/3.$

2.4 (*2 points*): *What is the value of* $P(A \cup B^c)$*?*

Answer: $P(A \cup B^c) = P(\{o_1\}) + P(\{o_2\}) + P(\{o_4\}) + P(\{o_5\}) = 1/2 + 1/3 + 1/18 + 1/18 = 17/18.$

2.5 (*2 points*): *What is the value of* $P(A \cap B|C)$*?*

Answer: $P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{0}{1/18+1/18} = 0.$

Question 3. (*10 points*)

Suppose you have spent the entire day studying in a windowless room in the library and it is now 7pm. When you left your house this morning, the forecast said there was a 20% chance it would be snowing at 7pm. Your friend is about to show up to help you study. You know from past experience that he wears a wooly hat with probability 0.9 when it is snowing but only wears a wooly hat with probability 0.3 when it is not snowing. Let H be the event that your friend is wearing the hat when he shows up and let S be the event that it is snowing outside.

3.1 (*4 points*): *Enter values for the following probabilities:*

$$P(S) = 0.2 \qquad P(H|S^c) = 0.3 \qquad P(H^c|S) = 0.1 \qquad P(H^c|S^c) = 0.7$$

3.2 (*2 points*): *What's the probability it is snowing and your friend is wearing his hat?*

Answer: $P(S \cap H) = P(S)P(H|S) = 0.2 \times 0.9 = 0.18 = 9/50.$

3.3 (*2 points*): *What's the probability your friend is wearing his hat?*

Answer: $P(H) = P(S)P(H|S) + P(S^c)P(H|S^c) = 0.18 + 0.8 \times 0.3 = 0.18 + 0.24 = 0.42 = 21/50.$

3.4 (*2 points*): *If your friend arrives and is not wearing his hat, what's the probability that it is snowing outside?*

Answer: $P(S|H^c) = P(S \cap H^c)/P(H^c) = P(S)P(H^c|S)/(1-0.42) = 0.2 \times 0.1/0.58 = 1/29.$

Question 4. (*10 points*) A branch of the sandwich shop *Subsetway* opens on campus. There are six sandwich fillings available:

$$\{\text{avocado, bacon, cheese, deli meat, egg, falafel}\} .$$

A popular option is to order the "Subsetway Special" which is a sandwich with three random different fillings and each subset of three fillings is equally likely. For example, you could get the set of fillings {avocado, bacon, cheese} or {bacon, egg, falafel} or {avocado, bacon, egg} etc.

4.1 (*2 points*): *How many different combinations of three fillings are there?*

Answer: $\binom{6}{3} = 20.$

4.2 (*2 points*): *How many different combinations of three fillings are there that include avocado?*

Answer: $\binom{5}{2} = 10.$

4.3 (*4 points*): *Let A be the event that your three fillings includes avocado and let B be the event that your three fillings include bacon. What are the values for the following probabilities:*

$$P(A) = \binom{5}{2}/\binom{6}{3} = 1/2$$

$$P(B) = \binom{5}{2}/\binom{6}{3} = 1/2$$

$$P(A \cap B) = 4/\binom{6}{3} = 1/5$$

Are A and B independent? *No since* $1/2 \times 1/2 \neq 1/5.$

4.4 (*2 points*): **Extra Credit:** *Another option that you could order is the "Subsetway Super Special" in which you get three fillings but this could be a triple helping of the same filling; or two of one filling and one of another; or three different fillings. If each possible sandwich is equally likely, what's the probability you get a triple helping of one fillings?*

Answer: There are 6 options where you get a triple helping. There are $2 \times \binom{6}{2} = 30$ options where you get two of one filling and one of another. There are $\binom{6}{3} = 20$ options where all three fillings are different. Hence the probability of getting a triple helping is $6/(6+30+20) = 5/56 = 3/28.$

Question 5. (*10 points*) You are bored. However, you have a six-sided dice and decide to solve some probability problems. Suppose you roll the dice five times. The set of possible outcomes Ω includes, for example, the sequences `11611`, `12345`, `13254` etc.

5.1 (*2 points*): *What is the size of the sample space?*

$$|\Omega| = 6^5 = 7776$$

5.2 (*2 points*): *Write out the outcomes in the event corresponding to getting the same value on each of the five rolls of the dice.*

Answer: `11111`, `22222`, `33333`, `44444`, `55555`, `66666`

5.3 (*2 points*): *What is the probability that you see five different values when you roll the dice five times?*

Answer: $\frac{6\times5\times4\times3\times2}{6^5} = 5/54.$

5.4 (*2 points*): *What is the probability that every value you observe during the five rolls is either* 1 *or* 2*? For example, seeing the sequence* `12122` *would be one outcome where every value is either* 1 *or* 2*.*

Answer: $\frac{2^5}{6^5} = 1/3^5 = 1/243.$

5.5 (*2 points*): *Which has a higher probability, the event that every value is 6 or the event that every number other than 6 appears exactly once during the 5 rolls. Justify your answer.*

Answer: The probability of getting five sixes is $\frac{1}{6^5}$ which is less than $\frac{5\times4\times3\times2\times1}{6^5}$, the probability that every number other than 6 appears exactly once.

Question 6. (*10 points*) Suppose you pick two cards randomly *without replacement* from a standard deck of cards. Recall that there are 52 cards and each card has one of four suits. There are 13 hearts, 13 clubs, 13 spaces, and 13 diamonds.

6.1 (*1 points*): *What's the probability that the cards are both clubs?*

Answer: $13/52 \times 12/51 = 3/51.$

6.2 (*2 points*): *What's the probability that neither card is clubs?*

Answer: $39/52 \times 38/51 = 19/34$.

6.3 (*2 points*): *What's the probability that the cards have different suits?*

Answer: $39/51$.

6.4 (*2 points*): *Let S be the event that exactly one card is spades and let C be the event that exactly one card is clubs. What are the values of the following probabilities.*

$$P(C) = 13/52 \times 39/51 + 39/52 \times 13/51 = 13/34$$

$$P(S) = 13/52 \times 39/51 + 39/52 \times 13/51 = 13/34$$

$$P(C \cap S) = 13/52 \times 13/51 + 13/52 \times 13/51 = 13/102$$

6.5 (*3 points*): *Let D be the event that the cards have different suits. What are the values of the following probabilities. For full marks you should simplify your answers fully.*

$$P(C|D) = \frac{6}{12} = 1/2$$

$$P(S|D) = \frac{6}{12} = 1/2$$

$$P(C \cap S|D) = \frac{2}{12} = 1/6$$

Are C and S independent conditioned on D? *No.*