Name: _______________________________ ID: ____________________

Instructions:

• Answer the questions directly on the exam pages.

• Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.

• Unless the question specifies otherwise, you may give your answer using arithmetic operations, such as addition, multiplication, “choose” notation and factorials (e.g., “9 \times 35! + 2” or “0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)” is fine).

• If you need extra space, use the back of a page.

• No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.

• If you have questions during the exam, raise your hand.

• The formulas for some standard random variables can be found on the last page.

<table>
<thead>
<tr>
<th>Question</th>
<th>Value</th>
<th>Points Earned</th>
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<tbody>
<tr>
<td>1</td>
<td>10</td>
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<td>10</td>
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<td>6</td>
<td>8+2 (Extra Credit)</td>
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<td>Total</td>
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**Question 1.** (10 points) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (2 points): $P(X = 0) = 0$ for every random variable $X$.

1.2 (2 points): For any non-negative random variable $X$, $P(X \geq 5) \leq E(X)/5$.

1.3 (2 points): For any random variable $X$, $P(X < E(X)) = P(X > E(X))$.

1.4 (2 points): $\sum_{k=1}^{\infty} (1 - p)^{k-1} p = 1$ for all values of $0 < p < 1$.

1.5 (2 points): The variance of a Bernoulli random variable is never larger than $1/4$.
Question 2. (10 points) Suppose that $X$ is a random variable where:

$$P(X = -1) = \frac{1}{4} \quad P(X = 0) = \frac{1}{2} \quad P(X = 1) = \frac{1}{4}$$

and let $Y = X^2 + 1$.

2.1 (2 points): What are the values of the following quantities:

$P(Y = 2) = \quad P(X = 0, Y = 2) =$

2.2 (2 points): What are the values of the following quantities:

$E(X) = \quad var(X) =$

2.3 (2 points): What are the values of the following quantities:

$E(Y) = \quad var(Y) =$

2.4 (2 points): What is the value of $P(X = 1|Y = 2)$?

2.5 (2 points): What is the value of $P(X \geq Y/3)$?
**Question 3.** (10 points) Suppose there is a baby who doesn’t sleep through the night. He sometimes has a bath before bedtime and sometimes he doesn’t. Let $B = 1$ if he has a bath and $B = 0$ if he doesn’t. Let $S$ be the number of hours he sleeps before waking up. Suppose $S$ is either 1, 2, or 3. The joint probabilities of $B$ and $S$ are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$S = 1$</th>
<th>$S = 2$</th>
<th>$S = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B = 0$</td>
<td>0.2</td>
<td>0.1</td>
<td>?</td>
</tr>
<tr>
<td>$B = 1$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.4</td>
</tr>
</tbody>
</table>

So, for example, $P(B = 0, S = 1) = 0.2$.

3.1 (1 points): What’s the value of the missing entry corresponding to $P(B = 0, S = 3)$?

$$P(B = 0, S = 3) =$$

3.2 (2 points): Enter the values for the following probabilities:

$$P(B = 0) =$$

$$P(B = 1) =$$

$$P(S = 1) =$$

$$P(S = 2) =$$

3.3 (2 points): Are $B$ and $S$ independent? Justify your answer.

3.4 (4 points): What are the values of the following quantities?

$$E(B) =$$

$$E(S) =$$

$$E(BS) =$$

$$	ext{cov}(B, S) =$$

3.5 (1 points): Based on the information given, does it seem reasonable to deduce that giving the baby a bath before bedtime increases the length of time until he wakes up? Justify your answer.
Question 4. (10 points) We say a random variable $X$ is memoryless if for all $a, b \geq 0,$
$$P(X > a + b|X > a) = P(X > b).$$

Let $X$ be a geometric random variable with parameter $p,$ i.e., $P(X = k) = (1 - p)^{k-1}p.$ You can use the formula $1 + x + x^2 + \ldots + x^j = \frac{x^{j+1} - 1}{x - 1}$ in this question but it is not necessary to do so.

4.1 (2 points): Express the probability $P(X > k)$ in terms of $k$ and $p.$ To get full marks you should simplify your answer fully.

4.2 (2 points): Show that $X$ is memoryless, i.e., $P(X > a + b|X > a) = P(X > b)$ for all $a, b \geq 0.$

4.3 (2 points): Are the events $\{X > a + b\}$ and $\{X > a\}$ independent? Justify your answer.

4.4 (2 points): Let $Y$ be a Poisson random variable with parameter $\lambda = 1.$ What are the values:
$$P(Y > 0) = \quad P(Y > 1) =$$

4.5 (2 points): Is $Y$ a memoryless random variable? Justify your answer. Recall that $e = 2.718 \ldots$
Question 5. (10 points) Consider tossing two fair coins, i.e., each coin is heads with probability 1/2. You may assume that the outcome of each coin is independent. Define the following four random variables:

\[ A = \begin{cases} 
1 & \text{if the first coin is Heads} \\
-1 & \text{if the first coin is Tails} 
\end{cases} \quad B = \begin{cases} 
1 & \text{if the second coin is Heads} \\
-1 & \text{if the second coin is Tails} 
\end{cases} \]

\[ C = A + B \quad D = \frac{A}{B} \]

5.1 (2 points): What are the following expected values:

\[ E(A) = \quad E(B) = \quad E(C) = \quad E(D) = \]

5.2 (2 points): What are the following variances:

\[ \text{var}(A) = \quad \text{var}(B) = \quad \text{var}(C) = \quad \text{var}(D) = \]

5.3 (2 points): Are \( A \) and \( C \) independent? Justify your answer.

5.4 (2 points): Are \( A \) and \( D \) independent? Justify your answer.

5.5 (2 points): Are \( A \) and \( A^2 \) independent? Justify your answer.
Question 6.  (10 points) You and a friend are playing a game. At each step you toss an unbiased coin. If the coin shows heads, your friend gives you $1. But if the coin shows tails, you give your friend $1. You repeat this game 100 times and therefore your final “winnings” will be somewhere between −$100 and $100. Hint: If \( W \) is your winnings then \( W = 2X - 100 \) where \( X \) is a binomial random variable corresponding to the number of heads observed.

6.1 (2 points): What’s the expected value and variance of your winnings?

6.2 (2 points): What’s the probability that your winnings are exactly $100?

6.3 (2 points): What’s the probability that your winnings are exactly $0?

6.4 (2 points): Use the Chebyshev bound to show an upper bound on the probability that you win or lose at least $20.
6.5 (2 points): **Extra Credit:** Suppose you repeat the game \( n \) times instead of 100 times. We say that you had a long winning streak if at some point during the game the coin shows heads \( k \) times in a row where \( k = 2 \log_2 n \). Show that the probability you had a winning streak is at most \( 1/n \). Hint: For \( i = 1, 2, 3 \ldots \) consider the event \( A_i \) corresponding to the \( i \)th, \((i + 1)\)th, \ldots, and \((i + k - 1)\)th coin tosses being heads.
Standard Random Variables

- Bernoulli Random Variable with parameter $p \in [0, 1]$:

  $$P(X = k) = \begin{cases} 
  1 - p & \text{if } k = 0 \\
  p & \text{if } k = 1
  \end{cases}, \quad E(X) = p, \quad var(X) = p(1 - p)$$

- Binomial Random Variable with parameters $p \in [0, 1]$ and $N \in \{1, 2, 3, \ldots\}$:

  For $k \in \{0, 1, 2, \ldots, N\}$: $P(X = k) = \binom{N}{k} p^k (1-p)^{N-k}, \quad E(X) = Np, \quad var(X) = Np(1-p)$

- Geometric Random Variable with parameter $p \in [0, 1]$:

  For $k \in \{1, 2, 3, \ldots\}$: $P(X = k) = (1-p)^{k-1} \cdot p, \quad E(X) = \frac{1}{p}, \quad var(X) = (1-p)/p^2$

- Poisson Random Variable with parameter $\lambda > 0$:

  For $k \in \{0, 1, 2, \ldots\}$: $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad E(X) = \lambda, \quad var(X) = \lambda$

- Discrete Uniform Random Variable with parameters $a, b \in \mathbb{Z}$ and $a < b$:

  For $k \in \{a, a+1, \ldots b\}$: $P(X = k) = \frac{1}{b-a+1}, \quad E(X) = \frac{a+b}{2}, \quad var(X) = \frac{(b-a+1)^2 - 1}{12}$