CMPSCI 240: Reasoning Under Uncertainty
First Midterm Exam

February 18, 2015.

Name: ______________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	8+2 (Extra Credit)	
Total	58	

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *If A and B are disjoint events then $P(A \cup B) = P(A) + P(B)$.*

Answer: TRUE.

1.2 (*2 points*): *If S is a set containing exactly 5 elements, then the power-set of S contains exactly 25 elements.*

Answer: FALSE. (The power-set contains $2^5 = 32$ elements.)

1.3 (*2 points*): *For any two events A and B, $P(A \cap B) = P(A)P(B)$.*

Answer: FALSE. (This is only true if A and B are independent.)

1.4 (*2 points*): *For any two events A and B where $0 < P(B) < 1$, $P(A|B) = 1 - P(A^c|B^c)$.*

Answer: FALSE. (It's true that $P(A|B) = 1 - P(A^c|B)$.)

1.5 (*2 points*): *If $P(A) > 1/2$ and $P(B) > 1/2$ then $P(A \cap B) > 0$.*

Answer: TRUE.

Question 2. (*10 points*) Babies are hungry 30% of the time. When a baby is hungry, they cry with probability 0.8. When a baby is not hungry, they cry with probability 0.5. Let H be the event that a baby is hungry and let C be the event that they are crying.

2.1 (*4 points*): *Enter values for the following probabilities:*

$$P(H) = 0.3 \qquad P(C|H) = 0.8 \qquad P(C^c|H) = 0.2 \qquad P(C|H^c) = 0.5$$

2.2 (*2 points*): *What's the probability the baby is crying and hungry?*

Answer: $P(C \cap H) = P(H)P(C|H) = 0.3 \times 0.8 = 0.24$.

2.3 (*2 points*): *What's the probability the baby is crying?*

Answer: $P(C) = P(H)P(C|H) + P(H^c)P(C|H^c) = 0.24 + 0.7 \times 0.5 = 0.59$.

2.4 (*2 points*): *If we can hear the baby crying, what's the probability that the baby is hungry?*

Answer: $P(H|C) = P(H \cap C)/P(C) = 0.24/0.59 = 24/59$.

Question 3. (*10 points*) Most debit cards have a 4-digit pin number. There are 10000 possible numbers:

$$\Omega = \{\texttt{0000}, \texttt{0001}, \texttt{0002}, \ldots, \texttt{9998}, \texttt{9999}\}$$

3.1 (*2 points*): *How many pin numbers are there where every digit is odd? Examples include* `1353` *and* `7777`.

Answer: $5^4 = 625$

3.2 (*2 points*): *How many pin numbers are there where at least one digit is even? Examples include* `1354` *and* `6768`.

Answer: $10000 - 625 = 9375$.

3.3 (*2 points*): *How many pin numbers are there where every digit is different? Examples include* `1354` *and* `6745`.

Answer: $10 \times 9 \times 8 \times 7 = 5040$.

3.4 (*2 points*): *How many pin numbers are there that are palindromes, i.e., they are the same when read forwards and backwards. Examples include* `1331` *and* `6776`.

Answer: $10^2 = 100$ since the first two digits could be any of 10 numbers and once you've picked the first two digits, the other two digits are fixed.

3.5 (*2 points*): *How many pin numbers are there where every digit is strictly larger than the preceding digit? Examples include* `1356` *and* `2459`.

Answer: $\binom{10}{4} = \binom{10}{6} = 210$. The simplest was to argue this is to start with the sequence `0123456789` and argue that you get a suitable pin number by deleting any subset of 6 characters. Hence there are $\binom{10}{6} = \binom{10}{4}$ possible sequences.

Question 4. (*10 points*) Draw two cards from a standard pack of cards *without replacement*. Recall that there are 52 cards and each card has one of four suits (hearts, clubs, spades, diamonds) and one of thirteen ranks (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king). Let A_1 be the event that the *first* card you draw is an ace and let K_2 be the event that the *second* card you draw is a king. Let D be the event that the two cards are from different suits.

4.1 (*4 points*): *Enter values for the following probabilities:*

$$P(A_1) = 1/13$$

$$P(K_2) = 1/13$$

$$P(A_1 \cap K_2) = P(A_1)P(K_2|A_1) = 1/13 \times 4/51$$

$$P(A_1^c \cap K_2) = P(K_2)P(A_1^c|K_2) = 1/13 \times 47/51$$

Answer: There are two ways of working about $P(K_2) = |K_2|/|\Omega|$. Either compute $|\Omega| = 52 \times 51$ (i.e., 52 choices for the first card and then 51 choices for the second card) and $|K_2| = 4 \times 51$ (i.e., 4 choices for the second card and then 51 choices for the first card). Or just argue that if you don't look at the first card, the second is equally likely to be any of the 52, or which 4 are a king.

4.2 (*1 points*): *Are the events A_1 and K_2 independent?*

Answer: No since $P(A_1) \times P(K_2) \neq P(A_1 \cap K_2)$.

4.3 (*4 points*): *Enter values for the following probabilities:*

$$P(A_1|D) = 1/13 \qquad P(K_2|D) = 1/13 \qquad P(A_1 \cap K_2|D) = 1/13^2 \qquad P(D) = 39/51$$

Answer: Follows from $|D| = 52 \times 39$, $|A_1 \cap D| = 4 \times 39$, $|K_2 \cap D| = 4 \times 39$, and $|(A_1 \cap K_2) \cap D| = 12$.

4.4 (*1 points*): *Are the events A_1 and K_2 independent conditioned on the event D?*

Answer: Yes since $P(A_1|D) \times P(K_2|D) = P(A_1 \cap K_2|D)$.

Question 5. (*10 points*) The "Dicey Die" Dice Company has manufactured a loaded dice. There are six possible outcomes $\{1, 2, 3, 4, 5, 6\}$ and the probabilities of these outcomes are:

$$P(1) = 0.1 \ , \ P(2) = 0.1 \ , \ P(3) = 0.1 \ , \ P(4) = 0.1 \ , \ P(5) = 0.1 \ , \quad \text{and} \quad P(6) = 0.5 \ .$$

5.1 (*4 points*): *For the experiment where you roll the loaded dice once, define the events $A = \{2, 4, 6\}$ and $B = \{3, 6\}$. Enter values for the following probabilities:*

$$\begin{aligned} P(A) &= 0.1 + 0.1 + 0.5 = 0.7 \\ P(B) &= 0.1 + 0.5 = 0.6 \\ P(A \cup B) &= 0.1 + 0.1 + 0.1 + 0.5 = 0.8 \\ P(A \cap B) &= 0.5 \end{aligned}$$

5.2 (*2 points*): *For the experiment where you roll the loaded dice three times, what's the probability you get exactly two sixes?*

Answer: The probability you get six-six-"not six" is $1/2^3$. The probability you get six-"not six"-six is also $1/2^3$. The probability you get "not six"-six-six is also $1/2^3$. Hence, the total probability is $3/8$.

5.3 (*2 points*): *For the experiment where you roll the loaded dice twice, what's the probability you get the same value both times?*

Answer: The probabilities of getting one-one, two-two, three-three, four-four, and five-five are each $0.1^2 = 0.01$. The probability of getting six-six is $0.5^2 = 0.25$. Hence, the total probability is $0.25 + 0.05 = 0.3$.

5.4 (*2 points*): *Suppose you roll a normal unloaded dice and the loaded dice at the same time. What's the probability you get the same value on both dice?*

Answer: The probability of getting one-one, two-two, three-three, four-four, and five-five is each $0.1 \times 1/6$. The probability of getting six-six is $0.5 \times 1/6$. Hence, the total probability is $0.5/6 + 0.5/6 = 1/6$. A simpler way to reason the answer is $1/6$ is to think that no mater the value of the loaded dice, the unloaded dice will be equal with probability $1/6$.

Question 6. (*10 points*) I am thinking of a random number in the range $\{1, 2, \ldots, 300\}$ and I am equally likely to be thinking of any of these three hundred numbers. Define the following events:

- A = "I'm thinking of a number that is divisible by two."
- B = "I'm thinking of a number that is divisible by three."
- C = "I'm thinking of a number that is divisible by five."

For example, $C = \{5, 10, 15, \ldots, 295, 300\}$, $|C| = 300/5 = 60$, and $P(C) = 1/5$.

6.1 (*4 points*): *Enter values for the following probabilities:*

$$\begin{aligned} P(A) &= 1/2 \\ P(B) &= 1/3 \\ P(A \cap B) &= 1/6 \\ P(A \cap B \cap C) &= 1/30 \end{aligned}$$

6.2 (*2 points*): *Enter values for the following probabilities:*

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = 1/2 + 1/3 - 1/6 = 2/3 \\ P(A^c \cap B^c) &= 1 - 2/3 = 1/3 \end{aligned}$$

6.3 (*2 points*): *What's the probability that the number I'm thinking of is divisible by exactly two of the numbers 2, 3, and 5?*

Answer: The probability of being divisible by 2 and 3 but not 5 is the probability of being divisible by and 3 minus the probability of being divisible by all three, i.e., $1/6 - 1/30$. Similarly the probability of being divisible by 3 and 5 but not 2 is $1/15 - 1/30$ and the probability of being divisible by 2 and 5 but not 3 is $1/10 - 1/30$. Hence the total probability is $1/6 + 1/15 + 1/10 - 1/10 = 7/30$.

6.4 (*2 points*): **Extra Credit:** *Define the additional event*

- D = *"I'm thinking of a number that is divisible by seven"*

and suppose I narrow the range of numbers I am thinking about to $\{1, 2, \ldots, 100\}$. *Write the event "I'm thinking of a number that isn't prime" in terms of the events* A, B, C *and* D. *What's the probability of this event?*

Answer: The event that the number I'm thinking of isn't a prime is $(A \cup B \cup C \cup D \cup \{1\}) \cap \{2, 3, 5, 7\}^c$. Note that:

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ &\quad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D| \\ &= 50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 + 0 - 0 = 78 \end{aligned}$$

Therefore, there are $78 + 1 - 4 = 75$ numbers less than 100 that aren't prime and the probability I pick a number that isn't prime is therefore $3/4$.