CMPSCI 240: Reasoning Under Uncertainty
Third Midterm Exam

April 16, 2014.

Name: ______________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	7	
3	9+3	
4	7	
5	10	
6	12+2	
Total	56+4	

Question 1. (*10 points*) Indicate whether each of the following five statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *For any random variables X, Y, and Z and values a, b, and c,*

$$P(X = a)P(Y = b|X = a)P(Z = c|X = a, Y = b) = P(Y = b)P(Z = c|Y = b)P(X = a|Y = b, Z = c)\ .$$

Answer: TRUE.

1.2 (*2 points*): *Suppose you throw 3 balls and each ball is equally likely to land in any one of 10 bins, then the probability that all balls land in the same bin is* $1/100$.

Answer: TRUE.

1.3 (*2 points*): *For any random variable X, $E(X^2) = E(X) \times E(X)$.*

Answer: FALSE. Since $var(X) = E(X^2) - E(X) \times E(X)$ the above statement would be equivalent to saying the variance is always 0.

1.4 (*2 points*): *For any two events A and B, $P(A^c \cup B^c) \leq 2 - P(A) - P(B)$.*

Answer: TRUE. By the union bound $P(A^c \cup B^c) \leq P(A^c \cup B^c) = 1 - P(A) + 1 - P(B) = 2 - P(A) - P(B)$.

1.5 (*2 points*): *If A and B are partitioning events, then $P(C|A) + P(C|B) = 1$ for any event C.*

Answer: FALSE. For example, if $C = \Omega$ then $P(C|A) + P(C|B) = 1 + 1 = 2$.

Question 2. (*7 points*) You're about to try a new type of pizza and you have two hypotheses:

$$H_1 = \text{ you will like the pizza} \qquad \text{and} \qquad H_2 = \text{ you won't like the pizza}$$

Suppose that you've tried 50 different types of pizza in your life and so far you've found 20 types that you liked and 30 that you didn't like. You also remember that:

5 of the types you liked included anchovies
10 of the types you liked included bacon
15 of the types you didn't like included anchovies
20 of the types you didn't like included bacon

Let A and B be the events the new type of pizza includes anchovies and bacon respectively. You may assume that $P(A \cap B|H_1) = P(A|H_1)P(B|H_1)$ and $P(A \cap B|H_2) = P(A|H_2)P(B|H_2)$.

2.1 (*1 points*): *What values should you use for the priors:*

$$P(H_1) = 2/5 \qquad P(H_2) = 3/5$$

2.2 (*2 points*): *What values should you use for the likelihoods:*

$$P(A|H_1) = 1/4 \qquad P(B|H_1) = 1/2 \qquad P(A|H_2) = 1/2 \qquad P(B|H_2) = 2/3$$

2.3 (*2 points*): *Which is the maximum a posteri (MAP) hypothesis. Show your working.*

Answer:

$$P(H_1|A \cap B) = \frac{P(H_1)P(A \cap B|H_1)}{P(A \cap B)} = \frac{P(H_1)P(A|H_1)P(B|H_1)}{P(A \cap B)} = \frac{2/5 \times 1/4 \times 1/2}{P(A \cap B)} = \frac{1/20}{P(A \cap B)}$$

$$P(H_2|A \cap B) = \frac{P(H_2)P(A \cap B)|H_2)}{P(A \cap B)} = \frac{P(H_2)P(A|H_2)P(B|H_2)}{P(A \cap B)} = \frac{3/5 \times 1/2 \times 2/3}{P(A \cap B)} = \frac{1/5}{P(A \cap B)}$$

So the MAP hypothesis is H_2.

2.4 (*2 points*): *What's the value of $P(A \cap B)$?*

Answer:

$$P(A \cap B) = P(H_1)P(A|H_1)P(B|H_1) + P(H_2)P(A|H_2)P(B|H_2) = 1/20 + 1/5 = 1/4.$$

Question 3. (*12 points*) A student is trying a new study strategy for the final exam. There are four topics to study for the exam and each day he picks a topic at random to study (and so he may study the same topic multiple times). Let T be the random variable corresponding to the number of days he'll study before he has studied every topic.

3.1 (*1 points*): *What are the values of the following probabilities?*

$$P(T=3) = 0 \qquad\qquad P(T=4) = 3/4 \times 2/4 \times 1/4 = 3/32$$

3.2 (*4 points*): *What are the expectation and variance of T?*

$$E(T) = 4/4 + 4/3 + 4/2 + 4/1 = 25/3$$

$$Var(T) = (1-1)/1^2 + (1-3/4)/(3/4)^2 + (1-2/4)/(2/4)^2 + (1-1/4)/(1/4)^2 = 130/9$$

For the next part of the question, suppose that the student only has 20 days to study. Let A_1 be the event that he does not study the first topic during these 20 days. Similarly, let A_2, A_3, and A_4 be the events that he doesn't study the second, third, and fourth topics during these 20 days.

3.3 (*2 points*): *What is the value of the following probability?*

$$P(A_1) = (3/4)^{20}$$

3.4 (*2 points*): *Use the union bound to show an upper bound for the following probability.*

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) \leq P(A_1) + P(A_2) + P(A_3) + P(A_4) = 4 \times (3/4)^{20}$$

3.5 (*3 points*): **Extra Credit:** *What is the value of the following probability?*

$$P(T=5) =$$

Answer: $T=5$ if and only if there is exact one day amongst the first five days that you study a subject that you've already studied. That day must be the second, third, or fourth day (it can't be the fifth because then $T=4$ since you'd have already studied all the topics after four days.)

$$\begin{aligned} P(T=5) &= 1/4 \times 3/4 \times 2/4 \times 1/4 \\ &\quad + 3/4 \times 2/4 \times 2/4 \times 1/4 \\ &\quad + 3/4 \times 2/4 \times 3/4 \times 1/4 \\ &= 9/64 \end{aligned}$$

Question 4. (*7 points*) A new phone comes on the market that claims to use extra strength glass. But you've watched some Youtube videos that suggest that the glass is just normal glass and you're curious if this is true. You have two hypotheses:

$$H_1 = \text{“phone manufacturer uses extra strength glass”}$$

$$H_2 = \text{“phone manufacturer uses normal glass”}$$

Your priors for these hypotheses are $P(H_1) = 1/10$ and $P(H_2) = 9/10$. To pick an hypothesis, you design the following experiment: you buy three identical phones and drop each from shoulder height. If the manufacturer is using extra strength glass, then each phone will break with probability 1/3. If the manufacturer is using normal glass, then each phone will break with probability 2/3. When you perform the experiment you observe that none of the three phones break. Call this event D.

4.1 (*2 points*): *What's the probability of D assuming there's extra strength glass and assuming there's normal glass.*

$$P(D|H_1) = (2/3)^3 = 8/27 \qquad\qquad P(D|H_2) = (1/3)^3 = 1/27$$

4.2 (*2 points*): *Given that event D is observed, which is the maximum a posteri (MAP) hypothesis? Show your working.*

Answer:

$$P(H_1|D) = P(D|H_1)P(H_1)/P(D) = (8/27 \times 1/10)/P(D) = (8/270)/P(D)$$

$$P(H_2|D) = P(D|H_2)P(H_2)/P(D) = (1/27 \times 9/10)/P(D) = (9/270)/P(D)$$

and so H_2 is the MAP hypothesis.

4.3 (*2 points*): *Suppose you drop another identical phone and this one also doesn't break. Which is the MAP hypothesis now? Show your working.*

Answer: Let D' be even that all four phones didn't break. Then

$$P(H_1|D') = P(D'|H_1)P(H_1)/P(D') = (16/81 \times 1/10)/P(D') = (16/810)/P(D')$$

$$P(H_2|D') = P(D'|H_2)P(H_2)/P(D') = (1/81 \times 9/10)/P(D') = (9/810)/P(D')$$

and so H_1 is now the MAP hypothesis.

4.4 (*1 points*): *The above experiments are quite expensive since they involve buying multiple phones. Suggest a different experiment that only involves buying one phone.*

Answer: Repeatedly drop the same phone. The observed data would then correspond to the number of times the phone was dropped before it broke. (Other possible solutions were accepted.)

Question 5. (*10 points*) The good news is that tomorrow you might have an ice cream. The bad news is that you might get sunburnt. Let $X = 1$ if you have an ice cream and let $X = 0$ if you don't. Let $Y = 1$ if you get sunburnt and let $Y = 0$ if you don't. Suppose the joint distribution of X and Y are given by the following table:

	$Y = 0$	$Y = 1$
$X = 0$	$5/12$	$1/12$
$X = 1$	$1/3$	$1/6$

So, for example $P(X = 0, Y = 1) = 1/12$.

5.1 (*2 points*): *Enter the values for the following probabilities:*

$P(X = 0) = 1/2$ $\quad$ $P(X = 1) = 1/2$ $\quad$ $P(Y = 0) = 3/4$ $\quad$ $P(Y = 1) = 1/4$

5.2 (*2 points*): *Are X and Y independent? Justify your answer numerically.*

Answer: No. For example, $P(X = 0, Y = 1) = 1/12 \neq 1/2 \times 1/4 = P(X = 0)P(Y = 1)$.

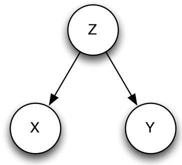
5.3 (*2 points*): *Let $Z = 1$ if it is sunny tomorrow and $Z = 0$ if it isn't. If $P(Z = 1) = 1/2$, $P(X = 1|Z = 1) = 2/3$, and $P(Y = 1|Z = 1) = 1/2$, what are the following values.*

$P(X = 0|Z = 1) = 1 - P(X = 1|Z = 1) = 1/3$ $\quad$ $P(Y = 0|Z = 1) = 1 - P(Y = 1|Z = 1) = 1/2$

5.4 (*2 points*): *What's the value of $P(Z = 1|X = 1)$?*

Answer: $P(Z = 1|X = 1) = \frac{P(X=1 \cap Z=1)}{P(X=1)} = \frac{P(X=1|Z=1)P(Z=1)}{P(X=1)} = \frac{2/3 \times 1/2}{1/2} = 2/3$.

5.5 (*2 points*): *What's the value of $P(X = 1, Y = 1, Z = 1)$ if the relationship between X, Y, and Z can be represented by the following Bayesian network:*



$$P(X = 1, Y = 1, Z = 1) = P(Z = 1)P(Y = 1|Z = 1)P(X = 1|Z = 1) = 1/2 \times 1/2 \times 2/3 = 1/6$$

Question 6. (*14 points*) Suppose you repeatedly toss a biased coin. Each time you toss the coin it lands heads with probability 1/3 and tails with probability 2/3. Let X be the random variable corresponding to the number of times you toss the coin until you've seen two heads. For example, when the sequence is of coin tosses is $HTTHHT\ldots$ then $X = 4$. Note that $X = X_1 + X_2$ where X_1 is the number of times you toss the coin until you see the first heads and X_2 is the number of additional tosses until you see the second heads.

6.1 (*2 points*): *What's the value of the expectation and variance of X?*

$E(X) = E(X_1) + E(X_2) = 3 + 3 = 6$ $\qquad Var(X) = Var(X_1) + Var(X_2) = \frac{1-1/3}{(1/3)^2} + \frac{1-1/3}{(1/3)^2} = 12$

6.2 (*2 points*): *What's the value of:*

$P(X = 2) = 1/3^2 = 1/9$ $\qquad P(X = 3) = 1/3 \times 2/3 \times 1/3 + 2/3 \times 1/3 \times 1/3 = 4/27$

6.3 (*2 points*): *Work out the formula for the following probability as a function of i:*

$$P(X = i) = (i-1)(1/3)^2(2/3)^{i-2} \quad \textit{for } i \geq 2$$

Answer: The factor $i - 1$ arises because the first heads can be in any of the first $i - 1$ positions.

6.4 (*2 points*): *Use the Markov bound to show an upper bound on $P(X \geq 12)$.*

Answer: $P(X \geq 12) \leq E(X)/12 = 6/12 = 1/2$

6.5 (*2 points*): *Use the Chebyshev bound to show an upper bound on $P(X \geq 12)$.*

Answer: $P(X \geq 12) = P(X \geq E(X) + 6) \leq P(|X - E(X)| \geq 6) \leq 12/6^2 = 1/3$.

6.6 (*2 points*): *What's the exact value of:*

$$P(X \geq 12 | X_1 = 2) = P(X_2 \geq 10) = (2/3)^9$$

6.7 (*2 points*): **Extra Credit:** *What's the exact value of $P(X \geq 12)$. To get the extra credit marks your answer should* not *involve an infinite sum. Hint: You may want to use the law of total probability and generalize your answer to the previous question.*

Answer:

$$\begin{aligned}
P(X \geq 12) = \sum_{i=1}^{\infty} P(X_1 = i)P(X_2 \geq 12 - i) &= \sum_{i=1}^{10} P(X_1 = i)P(X_2 \geq 12 - i) + \sum_{i=11}^{\infty} P(X_1 \geq i) \\
&= \sum_{i=1}^{10} (2/3)^{i-1}(1/3)(2/3)^{12-i-1} + (2/3)^{10} \\
&= 10 \times (1/3) \times (2/3)^{10} + (2/3)^{10} \\
&= \frac{13}{3} \times (2/3)^{10}
\end{aligned}$$

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

For $k \in \{0,1,2,\ldots,N\}$: $P(X=k) = \binom{N}{k} p^k \, (1-p)^{N-k}$, $E(X) = Np$, $var(X) = Np(1-p)$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

For $k \in \{1,2,3,\ldots\}$: $P(X=k) = (1-p)^{k-1} \cdot p$, $E(X) = \dfrac{1}{p}$, $var(X) = (1-p)/p^2$

- **Poisson Random Variable with parameter** $\lambda > 0$:

For $k \in \{0,1,2,\ldots\}$: $P(X=k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$, $E(X) = \lambda$, $var(X) = \lambda$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

For $k \in \{a, a+1, \ldots b\}$: $P(X=k) = \dfrac{1}{b-a+1}$, $E(X) = \dfrac{a+b}{2}$, $var(X) = \dfrac{(b-a+1)^2 - 1}{12}$

Bayes Formula

- If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^n P(B|A_j)P(A_j)}$$