# CMPSCI 240: Reasoning Under Uncertainty
# Second Midterm Exam

March 13, 2014.

Name: ______________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

| Question | Value | Points Earned |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 8 | |
| 5 | 8+2 (Extra Credit) | |
| 6 | 9+3 (Extra Credit) | |
| Total | 55 | |

**Question 1.** (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

**1.1** (*2 points*): *For every random variable* $X$, $P(X = 2) = 2 \times P(X = 1)$.

**1.2** (*2 points*): *For every random variable* $X$, $E(X^2) \geq E(X)^2$.

**1.3** (*2 points*): *If* $X$ *is a Poisson random variable, then* $E(X) = var(X)$. *Hint: See the last page of the exam for a formula sheet.*

**1.4** (*2 points*): *For every random variable that takes integer values, the expectation is an integer.*

**1.5** (*2 points*): *The variance of a geometric random variable is always larger than the expectation.*

**Question 2.** (*10 points*) Suppose that $X$ is a random variable where:

$$P(X = 1) = 1/2 \quad P(X = 2) = 1/4 \quad P(X = 4) = 1/4\ .$$

**2.1** (*2 points*): *What is the value of* $E(X)$*?*

**2.2** (*2 points*): *What is the value of* $E(X^2)$*?*

**2.3** (*2 points*): *What is the value of the standard deviation of* $X$*?*

**2.4** (*2 points*): *What is the value of* $P(X^2 - 3X + 2 = 0)$*?*

**2.5** (*2 points*): *Suppose* $Y$ *is another random variable that takes values from the set* $\{1, 2, 4\}$ *but the probabilities that it takes each value are unknown and some of them could be zero. What is the largest value that* $E(Y)$ *and* $var(Y)$ *can take?*

**Question 3.** (*10 points*) Your friend gives you a couple of coins that she claims are magic "quantum" coins and when you toss them at the same time something weird happens. Let $X = 1$ if the first coin is heads and let $X = 0$ if the first coin is tails. Let $Y = 1$ if the second coin is heads and let $Y = 0$ if the second coin is tails. The joint probabilities of $X$ and $Y$ are as follows:

| | $Y = 0$ | $Y = 1$ |
|---|---|---|
| $X = 0$ | 0.1 | 0.4 |
| $X = 1$ | 0.4 | 0.1 |

So, for example, $P(X = 0, Y = 1) = 0.4$.

**3.1** (*2 points*): *Enter the values for the following probabilities:*

$P(X = 0) =$ , $P(X = 1) =$ , $P(Y = 0) =$ , $P(Y = 1) =$

**3.2** (*2 points*): *Are $X$ and $Y$ independent? Justify your answer.*

**3.3** (*2 points*): *What's the PMF and expectation of $Z = X + Y$?*

$P(Z = 0) =$ , $P(Z = 1) =$ , $P(Z = 2) =$ , $E(Z) =$

**3.4** (*2 points*): *What's the value of the variance of $Z = X + Y$?*

**3.5** (*2 points*): *What's the value of the expectation of $XY$ and $cov(X, Y) = E(XY) - E(X)E(Y)$?*

**Question 4.** (*8 points*) Suppose we are conducting a poll to find out the percentage of Amherst residents that have one or more pets. When doing the poll, you may assume that we pick each person uniformly at random (so it's possible we might ask the same person more than once). Suppose we ask $t$ people and let $Q_i = 1$ if the $i$th person owns one or more pets and let $Q_i = 0$ if the $i$th person doesn't own a pet. Let $Q$ be the fraction of the people you poll that own pets, i.e.,

$$Q = \frac{Q_1 + Q_2 + \ldots + Q_t}{t} ,$$

and we consider $Q$ to be the "estimate" for the actual answer. Suppose 0.7 is the actual answer.

**4.1** (*2 points*): *Prove that $E(Q) = 0.7$ and that $var(Q) = 0.21/t$.*

**4.2** (*2 points*): *Use the Chebyshev bound to give a bound for $P(|Q - 0.7| \geq 0.01)$ in terms of $t$.*

**4.3** (*2 points*): *What's the smallest value of $t$ such that $P(0.69 < Q < 0.71) \geq 0.99$?*

**4.4** (*2 points*): *How would you change the poll if you wanted to estimate the average number of pets owned per person? You might need to ask more people to get a reasonably accurate answer. Why?*

**Question 5.** (*10 points*) Suppose you have a fair coin and keep tossing the coin until you see heads. Let $X$ be the number of times you toss the coin.

**5.1** (*2 points*): *What is the value of $E(X)$?*

**5.2** (*2 points*): *What is the value of $E(X^2)$? Hint: You might want to find the variance first.*

**5.3** (*2 points*): *What does the Markov bound imply about the value of $P(X \geq 3)$?*

**5.4** (*2 points*): *What is the* exact *value of $P(X \geq 3)$?*

**5.5** (*2 points*): **Extra Credit.** *Suppose you are competing with a friend who is tossing another fair coin and keeps tossing their coin until it lands heads. Let $Y$ be the number of times your friend tosses the coin. If $X$ and $Y$ are equal we consider it a draw. If $X > Y$ you win. What's the probability that there's a draw and what's the probability you win?*

**Question 6.** (*12 points*) Suppose that there are 730 people in the class and each person's birthday is equally likely to be any of the 365 days in the year (for this question you should ignore leap years). Define $X_1$ be the random variable corresponding to the number of people born on the first day of the year. Similarly, for $i = 2, 3, \dots, 365$, define $X_i$ to be the number of people born on the $i$th day of the year. So each $X_i$ takes a value in the range $\{0, 1, \dots, 730\}$ and, for example,

$$P(X_1 = 2) = \binom{730}{2} \times (1/365)^2 \times (364/365)^{728} ,$$

**6.1** (*3 points*): *Enter the values for the following quantities:*

$P(X_1 = 10) =$ $\qquad$ $E(X_1) =$ $\qquad$ $var(X_1) =$

**6.2** (*2 points*): *Let* $X = X_1 + X_2 + \ldots + X_{365}$. *Enter the values for the following quantities:*

$E(X) =$ $\qquad$ $var(X) =$

**6.3** (*2 points*): *What is the probability everyone has the same birthday, i.e.,* $\max(X_1, \dots, X_{365}) = 730$*?*

**6.4** (*2 points*): *What is the probability that everyone has different birthdays?*

**6.5** (*3 points*): **Extra Credit.** *Suppose it is the custom for each student to buy a present for everyone else in the class who shares his or her birthday. Let $Y$ be the total number of presents bought. What is the expected value of $Y$? Hint: First argue that there are $X_i(X_i - 1)$ presents bought for people who are born on the $i$th day of the year.*

$$E(Y) =$$

## Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

For $k \in \{0,1,2,\ldots,N\} : P(X=k) = \binom{N}{k} p^k \, (1-p)^{N-k}$ , $E(X) = Np$ , $var(X) = Np(1-p)$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

For $k \in \{1,2,3,\ldots\}$ : $P(X=k) = (1-p)^{k-1} \cdot p$ , $E(X) = \dfrac{1}{p}$ , $var(X) = (1-p)/p^2$

- **Poisson Random Variable with parameter** $\lambda > 0$:

For $k \in \{0,1,2,\ldots\}$ : $P(X=k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$ , $E(X) = \lambda$ , $var(X) = \lambda$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

For $k \in \{a, a+1, \ldots b\}$ : $P(X=k) = \dfrac{1}{b-a+1}$ , $E(X) = \dfrac{a+b}{2}$ , $var(X) = \dfrac{(b-a+1)^2 - 1}{12}$