CMPSCI 240: Reasoning Under Uncertainty
First Midterm Exam

February 19, 2014.

Name: ______________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	8+2 (Extra Credit)	
Total	58	

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *For any event A and sample space Ω,*

$$P(A \cup \Omega) = 1$$

Answer: TRUE

1.2 (*2 points*): *For any two events A and B where $0 < P(B) < 1$,*

$$P(A \cap B) \geq P(A|B)$$

Answer: FALSE. Consider $A = B$ and $P(A) = 1/2$. Then the left expression is $1/2$ but the right expression is 1.

1.3 (*2 points*): *For any two events A and B,*

$$A^c \cup B^c = (A \cap B)^c$$

Answer: TRUE.

1.4 (*2 points*): *For any three events A, B, and C where $P(A) > 0$ and $P(A \cap B) > 0$,*

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Answer: TRUE.

1.5 (*2 points*): *If A and B are disjoint events then A and B are independent.*

Answer: FALSE. Consider any disjoint sets A and B where $P(A) > 0$ and $P(B) > 0$. Then $P(A \cap B) = 0$ but $P(A)P(B) > 0$.

Question 2. (*10 points*) Suppose you roll a seven-sided dice to get an outcome from the set:

$$\Omega = \{1, 2, 3, 4, 5, 6, 7\} \ .$$

You should assume that each of the outcomes are equally likely. Consider the events

$$A = \text{"dice roll is odd"} \quad , \quad B = \text{"dice roll is even"} \quad , \quad C = \text{"dice roll is not 7"}$$

2.1 (*2 points*): *Enter values for the following probabilities:*

$$P(A) = 4/7 \qquad P(B) = 3/7 \qquad P(C) = 6/7$$

2.2 (*2 points*): *Enter values for the following probabilities:*

$$P(A \cap C) = 3/7 \qquad\qquad P(B \cap C) = 3/7$$

2.3 (*2 points*): *Enter values for the following probabilities. Write answer in the simplest form.*

$$P(A|C) = 1/2 \qquad\qquad P(B|C) = 1/2$$

2.4 (*1 points*): *Are the events A and C independent?*

Answer: No because $P(A) \neq P(A|C)$.

2.5 (*1 points*): *Are the events $A \cup B$ and C independent?*

Answer: Yes because $P(A \cup B) = 1, P(C) = 6/7$ and $P((A \cup B) \cap C) = 6/7 = 1 \times 6/7$.

2.6 (*2 points*): *If two events are independent, are the complements of the events always independent? Justify your answer.*

Answer: Yes. If A and B are independent, then

$$\begin{aligned} P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = 1 - P(A) - P(B) + P(A)P(B) &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A \cup B) \\ &= P((A \cup B)^c) \\ &= P(A^c \cap B^c) \end{aligned}$$

where we used the fact that $(A \cup B)^c = A^c \cap B^c$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Question 3. (*10 points*) Every afternoon I have a snack. Either I have a) an apple, b) a banana, c) chips, d) a donut, or e) empanadas. I choose my snack randomly but not all with the same probability. The sample space is $\Omega = \{a, b, c, d, e\}$ and the probability rule satisfies:

$$P(\{a\}) = 1/3 \quad , \quad P(\{b\}) = 1/6 \quad , \quad P(\{c\}) = 1/3 \quad , \quad P(\{d\}) = 1/12 \quad , \quad P(\{e\}) = 1/12$$

Let $F = \{a, b\}$ be the event that I pick a fruit. Let $V = \{c\}$ be the event that I had a snack that is available in the vending machine.

3.1 (*2 points*): *What is the value of $P(F^c)$?*

Answer: $P(F^c) = 1 - P(F) = 1 - P(\{a\}) - P(\{b\}) = 1 - 1/3 - 1/6 = 1/2$.

3.2 (*2 points*): *What is the value of $P(F \cup V)$?*

Answer: $P(F \cup V) = P(\{a\}) + P(\{b\}) + P(\{c\}) = 1/3 + 1/6 + 1/3 = 5/6$.

3.3 (*2 points*): *What's the probability I had an apple conditioned on the event that I had a fruit?*

Answer: $P(\{a\}|F) = P(\{a\} \cap F)/P(F) = P(\{a\})/P(F) = \frac{1/3}{1/3+1/6} = 2/3$.

3.4 (*1 points*): *What's the probability I eat a donut* or *empanadas?*

Answer: $P(\{d, e\}) = P(\{d\}) + P(\{e\}) = 1/12 + 1/12 = 1/6$.

3.5 (*1 points*): *What's the probability I eat a donut* and *empanadas?*

Answer: $P(\{d\} \cap \{e\}) = P(\emptyset) = 0$.

3.6 (*2 points*): *Find a partition of Ω into events such that each event has probability 1/2.*

Answer: The events $F = \{a, b\}$ and $F^c = \{c, d, e\}$ partition Ω such that $P(F) = P(F^c) = 1/2$.

Question 4. (*10 points*) Suppose that 1 in 5 emails I receive are spam. When my computer receives spam it puts it in the junk mail folder with probability 5/6. When my computer receives a message that isn't spam, it puts it in the junk mail folder with probability 1/3. Define the events:

$$S = \text{“email is spam”} \quad , \quad J = \text{“email gets put in my junk mail folder”}$$

4.1 (*2 points*): *Enter the values for the following probabilities:*

$P(S) = 1/5 \quad P(S^c) = 4/5 \quad P(J|S) = 5/6 \quad P(J|S^c) = 1/3 \quad P(J^c|S) = 1/6 \quad P(J^c|S^c) = 2/3$

4.2 (*2 points*): *What's the probability that the next email is spam and is put in the junk mail folder?*

Answer: $P(S \cap J) = P(S)P(J|S) = 1/5 \times 5/6 = 1/6$

4.3 (*2 points*): *What's the probability that the next email is not spam and is put in the junk mail folder?*

Answer: $P(S^c \cap J) = P(S^c)P(J|S^c) = 4/5 \times 1/3 = 4/15$

4.4 (*2 points*): *What's the probability that the next email gets put in the junk mail folder?*

Answer: By the law of total probability

$$P(J) = P(S)P(J|S) + P(S^c)P(J|S^c) = 1/5 \times 5/6 + 4/5 \times 1/3 = 1/6 + 4/15 = 13/30 \ .$$

4.5 (*2 points*): *What's the probability that an email in the junk mail folder is actually spam?*

Answer: $P(S|J) = P(S \cap J)/P(J) = \frac{1/6}{13/30} = 5/13.$

Question 5. (*10 points*) Suppose you are designing a new security system for the computer science building. The main door will have a keypad with ten keys: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. To gain entrance, the user needs to enter a code consisting of four numbers.

5.1 (*2 points*): *How many possible codes are there if the numbers are entered one at a time. You should assume that the order in which the numbers are entered matters (e.g., the code 1234 is different from the code 4321), and numbers can be used more than once (e.g., 4233 is a valid code).*

Answer: There are 10 choices for each number in the sequence so there are 10^4 sequences.

5.2 (*2 points*): *How many codes are there if the numbers are entered one at a time and the order matters, but no numbers can appear more than once (e.g., 1231 is not a valid code)?*

Answer: There are 10 choices for first number, 9 choices for the second number, 8 choices for the third number and 7 choices for the fourth number. Hence there are $10 \times 9 \times 8 \times 7$ choices in total.

5.3 (*2 points*): *How many codes are there if the numbers are entered one at a time and the order matters, but no two consecutive numbers are the same (e.g., 1231 is a valid code but 3112 is not)?*

Answer: There are 10 choices for first number, 9 choices for the second number (since it can't be the same as the previous number), 9 choices for the third number (since it can't be the same as the previous number), and 9 choices for the fourth number (since it can't be the same as the previous number). Hence there are $10 \times 9 \times 9 \times 9$ choices in total.

5.4 (*2 points*): *How many codes are there if the numbers are entered one at a time and the numbers need to be entered in strictly increasing order (e.g., 1234 is a valid code but 1243 is not)?*

Answer: There are $\binom{10}{4}$ choices since once you decide which subset of four numbers are in the code, there is only one choice for the order in which you would enter them.

5.5 (*2 points*): *How many codes are there if all four numbers need to be pressed simultaneously?*

Answer: There are $\binom{10}{4}$ choices since if you are pressing them simultaneously only the subset of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ you are pressing matters.

Question 6. (*10 points*) Suppose you pick two cards randomly *without replacement* from a standard deck of cards. Recall that there are 52 cards and each card has one of four suits. There are 13 hearts, 13 clubs, 13 spaces, and 13 diamonds.

6.1 (*2 points*): *What's the probability that the cards are both hearts?*

Answer: $\frac{13}{52} \times \frac{12}{51}$

6.2 (*2 points*): *What's the probability that both cards have the same suit?*

Answer: $\frac{12}{51}$ since, no matter what the first card was, there are 12 cards out of the remaining 51 that have the same suit.

6.3 (*2 points*): *What's the probability that the cards have different suits?*

Answer: $1 - \frac{12}{51} = \frac{39}{51}$ since this is the complement of the event that both cards have the same suit.

6.4 (*2 points*): *What's the probability that there are the same number of hearts and diamonds? Note that if there are zero of both, that still counts as the same number.*

Answer: The probability of having 0 hearts and 0 diamonds is $26/52 \times 25/51$. The probability of having 1 heart and 1 diamond is $2 \times 13/52 \times 13/51$ since you could pick hearts first and then diamonds or vice versa. Hence, the total probability is $26/52 \times 25/51 + 2 \times 13/52 \times 13/51 = 19/51$.

6.5 (*2 points*): **Extra Credit:** *Let S be the set of 52 cards in a standard deck. Let a be the number of subsets of size k. Let b be number of subsets of size k that include the ace of spades. Let c be number of subsets of size k that do not include the ace of spades. Using "choose" notation, what are the values of a, b, and c as a function of k.*

$$a = \binom{52}{k} \qquad b = \binom{51}{k-1} \qquad c = \binom{51}{k}$$

Write out a formula that relates a, b, and c. Your formula should not depend on k.

Answer: You can divide the set of subsets of size k into those that include the ace of spaces and those that don't. Hence $a = b + c$. More generally, this line of thinking allows you to prove the following equation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Note that this is the rule you use when you compute Pascal's triangle.