Name: ___________________________ ID: ___________________________

Instructions:

• Answer the questions directly on the exam pages.

• Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.

• If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., “$9 \times 35! + 2$” or “$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$” is fine) unless the problem says otherwise.

• If you need extra space, use the back of a page.

• No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.

• If you have questions during the exam, raise your hand.

• The formulas for some standard random variables can be found on the last page.

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Question 1. (10 points) Indicate whether each of the following five statements is TRUE or FALSE. No justification is required.

1.1 (2 points): The probability of an event is always at most 1.
Answer: TRUE.

1.2 (2 points): For any three events $A, B,$ and $C$ such that $P(A \cap B) > 0$ and $P(B \cap C) > 0$ then
$$P(A)P(B|A)P(C|A \cap B) = P(C)P(B|C)P(A|B \cap C) .$$
Answer: TRUE.

1.3 (2 points): If a Markov Chain is aperiodic and irreducible then there exists a distribution $v^*$ over states such that the distribution over states at time $t$ approaches $v^*$ as $t$ increases.
Answer: TRUE.

1.4 (2 points): Let $X$ be a binomial random variable with parameters $N=100$ and $p=1/2$. Then
$$P(X = 10) < P(X = 90) .$$
Answer: FALSE. For those parameters, $P(X = 10) = P(X = 90)$

1.5 (2 points): For any random variables $X$ and $Y$ where $Y = X + 10$, then $\text{var}(X) = \text{var}(Y) + 10$.
Answer: FALSE. For any constant $c$, $\text{var}(X + c) = \text{var}(X)$. 

Question 2. (10 points) Let $X$ be a random variable where

$$P(X = 1) = 1/3 \quad P(X = 2) = 1/3 \quad P(X = 3) = 1/3.$$ 

Let $Y = (X - 1)(X - 2)$ be another random variable that depends on $X$. To get full marks you need to simplify your answers to a single number.

2.1 (3 points): Compute the following quantities:

$$E(X) = 1/3 + 2/3 + 3/3 = 2 \quad E(X^2) = 1/3 + 2^2/3 + 3^2/3 = 14/3 \quad \text{var}(X) = 14/3 - 2^2 = 2/3$$

2.2 (3 points): Compute the following quantities:

$$P(Y = 0) = P(\{X = 1\} \cup \{X = 2\}) = 2/3 \quad P(Y = 2) = P(X = 3) = 1/3 \quad \text{var}(Y) = 8/9$$

2.3 (2 points): Compute the following quantities:

$$E(X + Y) = E(X) + E(Y) = 2 + 2/3 = 8/3 \quad E\left(\frac{Y}{X}\right) = P(X = 3) \times \frac{2}{3} = 2/9$$

2.4 (2 points): Recall that the covariance of $X$ and $Y$ is $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$. Compute the covariance of $X$ and $Y$.

$$\text{cov}(X,Y) = P(X = 3) \times 3 \times 2 - 2 \times 2/3 = 2 - 4/3 = 2/3$$
**Question 3.** (10 points) In this problem we consider throwing balls into bins. Suppose there are five bins and each time we throw a ball it is equally likely to land in any of the five bins. In the next three questions, suppose we throw exactly five balls.

3.1 (2 points): What’s the probability that all the balls land in different bins?

**Answer:** $\frac{5}{5} \times \frac{4}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5}$

3.2 (2 points): What’s the probability that all the balls land in the same bin?

**Answer:** $(\frac{1}{5})^4$

3.3 (2 points): What’s the probability that there is one bin with exactly three balls and one bin with exactly two balls?

**Answer:** Suppose the bins are labelled $A, B, C, D, E$ and consider the sequence corresponding to where the balls landed, e.g., $AABBB$ would be the sequence if the first two balls landed in bin $A$ and the next three landed in bin $B$. There are 5 choices for which bin contains 3 balls and 4 remaining choices for which bin contains 2 balls. Of the 5 balls that were thrown, there are $\binom{5}{3}$ choices for which 3 balls went in the bin that you’d designated to contain 3 balls. Hence there are $5 \times 4 \times \binom{5}{3} = 200$ sequences in which exactly 3 balls landed in one bin and 2 balls landed in another. Since, each sequence has probability $\frac{1}{5^5}$ the probability is $200 \times \frac{1}{5^5} = \frac{8}{125}$.

For the next two questions, suppose you throw balls until all the bins contain at least one ball and let $X$ be the total number of balls that you throw. Let $B_1, B_2, B_3, B_4,$ and $B_5$ be the number of balls in the first, second, third, fourth, and fifth bin when you’ve thrown the $X$ balls.

3.4 (2 points): What’s the expected value of $X$?

**Answer:** $\frac{5}{5} + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1}$

3.5 (2 points): What’s the value of $P(B_1 = 2 | X = 6)$?

**Answer:** If $X = 6$ then there is one bin with two balls and 4 bins with one ball. Since each ball is equally likely to land in each bin, there is a $\frac{1}{5}$ probability that the bin with two balls is the first bin.
Question 4. (10 points) Consider the Markov chain with two states \( \{s_1, s_2\} \) and transition matrix \( M \):

\[
M = \begin{pmatrix}
    1/2 & 1/2 \\
    1 & 0
\end{pmatrix}
\]

For example, if you are in state \( s_2 \) then the probability you would move to state \( s_1 \) is 1. Let \( X_t \) be the random variable corresponding to the state after \( t \) steps. The initial state is \( s_1 \), i.e., \( X_0 = s_1 \).

4.1 (2 points): Draw the transition diagram of the Markov chain and label each edge with the appropriate transition probability.

**Answer:**

![Transition Diagram](image)

4.2 (2 points): What’s the steady state distribution of the Markov chain?

**Answer:** Let \((a, 1 - a)\) be the steady state distribution. Then,

\[
(a, 1 - a) = (a, 1 - a) \begin{pmatrix}
    1/2 & 1/2 \\
    1 & 0
\end{pmatrix} = (1 - a/2, a/2)
\]

Therefore \(1 - a = a/2\) and so \(a = 2/3\). Hence the steady state distribution is \((2/3, 1/3)\).

4.3 (4 points): What’s the values of the following probabilities. In this question you need to fully simplify your answer to get full marks.

\[
\begin{align*}
P(X_1 = s_1) & = 1/2 \\
P(X_2 = s_1) & = 1/2^2 + 1/2 = 3/4 \\
P(X_3 = s_1) & = 1/2^3 + 2/2^2 = 5/8 \\
P(X_4 = s_1) & = 1/2^4 + 3/2^3 + 1/2^2 = 11/16
\end{align*}
\]

4.4 (2 points): Extra Credit: What’s the probability \(P(X_t = s_1)\) as a function of \(t\)? To get full marks your answer should be fully simplified. Hint: If the steady state distribution is \((\alpha, 1 - \alpha)\) for some \(0 < \alpha < 1\) consider what happens to \((\alpha + \gamma, 1 - \alpha - \gamma)\) for some arbitrary \(\gamma\).

**Answer:** Following the hint, first note that:

\[
(2/3 + \gamma, 1/3 - \gamma) \begin{pmatrix}
    1/2 & 1/2 \\
    1 & 0
\end{pmatrix} = (2/3 - \gamma/2, 1/3 + \gamma/2)
\]

Therefore, if the initial distribution is \((2/3 + 1/3, 1/3 - 1/3)\) then after \(t\) steps the distribution is \((2/3 + 1/3 \times (-1/2)^t, 1/3 - 1/3 \times (-1/2)^t)\) and so \(P(X_t = s_1) = 2/3 + 1/3 \times (-1/2)^t\).
Question 5. (10 points) Suppose you bought a light bulb. Let \( X \) be the random variable corresponding to the number of years that the light bulb lasts. You have two hypotheses for the distribution of \( X \) corresponding to how well the bulb was made:

\[
H_1 = \text{“} X \text{ is a geometric random variable with parameter } \frac{2}{3} \text{.”}
\]

\[
H_2 = \text{“} X \text{ is a geometric random variable with parameter } \frac{1}{2} \text{.”}
\]

Your priors for these hypotheses are \( P(H_1) = \frac{3}{5} \) and \( P(H_2) = \frac{2}{5} \).

5.1 (2 points): Let \( \{X = 1\} \) be the event the light bulb lasts for one year. What are the values:

\[
P(X = 1|H_1) = \frac{2}{3} \quad P(X = 1|H_2) = \frac{1}{2}
\]

\[
P(X = 3|H_1) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27} \quad P(X = 3|H_2) = \frac{1}{8}
\]

5.2 (2 points): If event \( \{X = 1\} \) is observed, which is the MAP hypothesis? Show your working.

Answer: The MAP hypothesis is \( H_1 \) because:

\[
P(H_1|X = 1) = P(X = 1|H_1)P(H_1)/P(X = 1) = \frac{2}{3} \times \frac{3}{5}/\frac{3}{5} = \frac{2}{5}/P(X = 1)
\]

\[
P(H_2|X = 1) = P(X = 1|H_2)P(H_2)/P(X = 1) = \frac{1}{2} \times \frac{2}{5}/P(X = 1) = \frac{1}{5}/P(X = 1)
\]

5.3 (2 points): If event \( \{X = 3\} \) is observed, which is the MAP hypothesis? Show your working.

Answer: The MAP hypothesis is \( H_2 \) because:

\[
P(H_1|X = 3) = P(X = 3|H_1)P(H_1)/P(X = 3) = \frac{2}{27} \times \frac{3}{5}/\frac{3}{5} = \frac{2}{45}/P(X = 3)
\]

\[
P(H_2|X = 3) = P(X = 3|H_2)P(H_2)/P(X = 3) = \frac{1}{8} \times \frac{2}{5}/P(X = 3) = \frac{1}{20}/P(X = 3)
\]

5.4 (2 points): What is the value of \( P(X = i) \) as a function of \( i \) based on the priors above?

Answer:

\[
P(X = i) = P(X = i|H_1)P(H_1) + P(X = i|H_2)P(H_2) = \left(\frac{1}{3}\right)^{i-1} \times \frac{2}{3} \times \frac{3}{5} + \left(\frac{1}{2}\right)^i \times \frac{2}{5}
\]

5.5 (2 points): What is the value of \( E(X) \)? To get full marks your answer should be fully simplified.

Answer: \( E(X) = \sum_{i=1}^{\infty} i \times \left(\frac{1}{3}\right)^{i-1} \times \frac{2}{3} \times \frac{3}{5} + \left(\frac{1}{2}\right)^i \times \frac{2}{5} \). Observe that this is \( \frac{3}{5} \) times the expectation of a geometric random variable with parameter \( \frac{2}{3} \) plus \( \frac{2}{5} \) times the expectation of a geometric random variable with parameter \( \frac{1}{2} \). Hence it equals \( \frac{3}{5} \times \frac{3}{2} + \frac{2}{5} \times 2 = \frac{17}{10} \).
Question 6.  (10 points)  Good news! The computer science vending machine has been updated to include some healthy snacks and Mexican favorites. The available options are:

\[ \Omega = \{ \text{Apple, Banana, Chips, Donut, Empanada} \} \]

The probabilities of picking these options are:

\[ P(\text{Apple}) = \frac{4}{10} \quad P(\text{Banana}) = \frac{2}{10} \quad P(\text{Chips}) = \frac{2}{10} \quad P(\text{Donut}) = \frac{1}{10} \quad P(\text{Empanada}) = \frac{1}{10} . \]

6.1 (2 points): What is the probability of eating a fruit? Hint: An empanada is not a fruit.

**Answer:** \( P(\text{a fruit}) = P(\text{Apple}) + P(\text{Banana}) = \frac{3}{5} \).

6.2 (2 points): Suppose that to pick one of the items you enter a binary sequence of length \( k \). What is the minimum value of \( k \) that can be used such that all items have a different binary sequence.

**Answer:** There are 8 sequences of length \( k = 3 \) and only 4 sequences of length \( k = 2 \). Hence \( k = 3 \) is the smallest value such that each item has a different binary sequence.

In the rest of this question, the sequences are of different lengths and are chosen according to the following tree, e.g., the sequence for “Banana” is 0 and the sequence for “Empanada” is 111.

![Tree Diagram]

6.3 (2 points): What are the sequences for:

- “Chips”  = 100
- “Apple”  = 101
- “Donut”  = 110

6.4 (2 points): If I pick an item according to the above probabilities, let \( X \) be the length of the corresponding sequence. What is the expected value of \( X \)?

**Answer:** \[ E(X) = P(\text{Banana}) \times 1 + (1 - P(\text{Banana})) \times 3 = \frac{2}{10} + \frac{8}{10} \times 3 = \frac{26}{10} = 2.6. \]

6.5 (2 points): Extra Credit: Find a tree in which the expected length of the sequence is smaller. What is the expected length in this case.
**Answer:** Consider the following encoding tree:

Then the expected length of a sequence is

\[
\frac{4}{10} \times 1 + \frac{2}{10} \times 2 + \frac{2}{10} \times 3 + \frac{1}{10} \times 4 + \frac{1}{10} \times 4 = \frac{11}{5} = 2.2
\]
Question 7. (10 points) It’s the day of the big game and outside the Mullins center there are two competing t-shirt vendors, Alice and Bob. They each paid $100 for a license to sell t-shirts and are each now deciding whether to sell their t-shirts for $10 or $20.

- If both sell their t-shirts at $10 then each will sell 10 t-shirts.
- If both sell their t-shirts at $20 then each will sell 5 t-shirts.
- If Alice sells at $10 and Bob sells at $20 then Alice sells 20 t-shirts and Bob sells none.
- If Bob sells at $10 and Alice sells at $20 then Bob sells 20 t-shirts and Alice sells none.

You may assume that the t-shirts were effectively free to make and so the license is the only cost.

7.1 (2 points): For each combination of strategies, what is Alice’s profit? Remember to include the cost of the license and so the profit could be negative.

- Alice’s profit if both sell at $10 = 10 \times 10 - 100 = 0
- Alice’s profit if both sell at $20 = 5 \times 20 - 100 = 0
- Alice’s profit if Alice sells at $10 and Bob sells at $20 = 20 \times 10 - 100 = 100
- Alice’s profit if Alice sells at $20 and Bob sells at $10 = -100

7.2 (1 points): Is this a zero-sum game?

Answer: Yes.

7.3 (2 points): Suppose Alice sets her price at $10 with probability p. Bob sets his price at $10 with probability q. As a function of p and q, what is the expected profit for Alice?

Answer: \( pq \times 0 + (1 - p)(1 - q) \times 0 + p(1 - q) \times 100 - (1 - p)q \times 100 = 100p - 100q. \)

7.4 (1 points): What value should Alice choose for p if q = 1/2.

Answer: If q = 1/2, Alice’s expected profit is 100p − 50 and this is increase by increasing p. Hence Alice would set p = 1.

7.5 (2 points): Find a Nash equilibrium for the game. (You just need to find one.)

Answer: If both players set their price to be $10 then neither has an incentive to change their strategy. Hence both players setting their price to be $10 is a Nash equilibrium.
7.6 (2 points): **Extra Credit:** Alice is considering bribing Bob to close his t-shirt stand so that she has no competition. Ignoring anti-competition laws and moral issues, what’s the maximum she should pay Bob? What assumptions are you making in your answer?

**Answer:** If Bob doesn’t close assume that Alice makes $0 profit but if Bob does close then Alice can make $100 (either by selling 20 t-shirts and $10 or selling 10 t-shirts at $20). Hence, Alice should be prepared to pay up to $100. This is the maximum she should pay Bob. However note that from Bob’s point of view, he’s making $0 profit if both stay open. Hence, if he’s only paid one cent he has an incentive to close.
Standard Random Variables

- **Bernoulli Random Variable** with parameter \( p \in [0, 1] \):
  
  \[
P(X = k) = \begin{cases} 
  1 - p & \text{if } k = 0 \\
  p & \text{if } k = 1 
\end{cases}, \quad E(X) = p, \quad \text{var}(X) = p(1 - p)
\]

- **Binomial Random Variable** with parameters \( p \in [0, 1] \) and \( N \in \{1, 2, 3, \ldots\} \):
  
  For \( k \in \{0, 1, 2, \ldots, N\} \) : 
  
  \[
P(X = k) = \binom{N}{k} p^k (1-p)^{N-k}, \quad E(X) = Np, \quad \text{var}(X) = Np(1-p)
\]

- **Geometric Random Variable** with parameter \( p \in [0, 1] \):
  
  For \( k \in \{1, 2, 3, \ldots\} \) : 
  
  \[
P(X = k) = (1-p)^{k-1} \cdot p, \quad E(X) = \frac{1}{p}, \quad \text{var}(X) = (1-p)/p^2
\]

- **Poisson Random Variable** with parameter \( \lambda > 0 \):
  
  For \( k \in \{0, 1, 2, \ldots\} \) : 
  
  \[
P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad E(X) = \lambda, \quad \text{var}(X) = \lambda
\]

- **Discrete Uniform Random Variable** with parameters \( a, b \in \mathbb{Z} \) and \( a < b \):
  
  For \( k \in \{a, a+1, \ldots, b\} \) : 
  
  \[
P(X = k) = \frac{1}{b - a + 1}, \quad E(X) = \frac{a + b}{2}, \quad \text{var}(X) = \frac{(b - a + 1)^2 - 1}{12}
\]

Bayes Formula

- If \( A_1, \ldots, A_n \) partition \( \Omega \) then for any event \( B \):
  
  \[
P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}
\]