CMPSCI 240: Reasoning Under Uncertainty
Final Exam

May 2, 2014.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine) unless the problem says otherwise.
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	8+2	
5	10	
6	8+2	
7	8+2	
Total	64+6	

Question 1. (*10 points*) Indicate whether each of the following five statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *The probability of an event is always at most 1.*

1.2 (*2 points*): *For any three events $A, B,$ and C such that $P(A \cap B) > 0$ and $P(B \cap C) > 0$ then*

$$P(A)P(B|A)P(C|A \cap B) = P(C)P(B|C)P(A|B \cap C) \ .$$

1.3 (*2 points*): *If a Markov Chain is aperiodic and irreducible then there exists a distribution v^* over states such that the distribution over states at time t approaches v^* as t increases.*

1.4 (*2 points*): *Let X be a binomial random variable with parameters $N = 100$ and $p = 1/2$. Then*

$$P(X = 10) < P(X = 90) \ .$$

1.5 (*2 points*): *For any random variables X and Y where $Y = X + 10$, then $var(X) = var(Y) + 10$.*

Question 2. (*10 points*) Let X be a random variable where

$$P(X = 1) = 1/3 \qquad P(X = 2) = 1/3 \qquad P(X = 3) = 1/3 \ .$$

Let $Y = (X - 1)(X - 2)$ be another random variable that depends on X. To get full marks you need to simplify your answers to a single number.

2.1 (*3 points*): *Compute the following quantities:*

$$E(X) = \qquad\qquad E(X^2) = \qquad\qquad var(X) =$$

2.2 (*3 points*): *Compute the following quantities:*

$$P(Y = 0) = \qquad\qquad P(Y = 2) = \qquad\qquad var(Y) =$$

2.3 (*2 points*): *Compute the following quantities:*

$$E(X + Y) = \qquad\qquad E\left(\frac{Y}{X}\right) =$$

2.4 (*2 points*): *Recall that the covariance of X and Y is $cov(X, Y) = E(XY) - E(X)E(Y)$. Compute the covariance of X and Y.*

$$cov(X, Y) =$$

Question 3. (*10 points*) In this problem we consider throwing balls into bins. Suppose there are five bins and each time we throw a ball it is equally likely to land in any of the five bins. In the next three questions, suppose we throw exactly five balls.

3.1 (*2 points*): *What's the probability that all the balls land in different bins?*

3.2 (*2 points*): *What's the probability that all the balls land in the same bin?*

3.3 (*2 points*): *What's the probability that there is one bin with exactly three balls and one bin with exactly two balls?*

For the next two questions, suppose you throw balls until all the bins contain at least one ball and let X be the total number of balls that you throw. Let B_1, B_2, B_3, B_4, and B_5 be the number of balls in the first, second, third, fourth, and fifth bin when you've thrown the X balls.

3.4 (*2 points*): *What's the expected value of X?*

3.5 (*2 points*): *What's the value of $P(B_1 = 2|X = 6)$?*

Question 4. (*10 points*) Consider the Markov chain with two states $\{s_1, s_2\}$ and transition matrix M:

$$M = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$$

For example, if you are in state s_2 then the probability you would move to state s_1 is 1. Let X_t be the random variable corresponding to the state after t steps. The initial state is s_1, i.e., $X_0 = s_1$.

4.1 (*2 points*): *Draw the transition diagram of the Markov chain and label each edge with the appropriate transition probability.*

4.2 (*2 points*): *What's the steady state distribution of the Markov chain?*

4.3 (*4 points*): *What's the values of the following probabilities. In this question you need to fully simplify your answer to get full marks.*

$$P(X_1 = s_1) \quad =$$

$$P(X_2 = s_1) \quad =$$

$$P(X_3 = s_1) \quad =$$

$$P(X_4 = s_1) \quad =$$

4.4 (*2 points*): **Extra Credit:** *What's the probability* $P(X_t = s_1)$ *as a function of* t*? To get full marks your answer should be fully simplified. Hint: If the steady state distribution is* $(\alpha, 1-\alpha)$ *for some* $0 < \alpha < 1$ *consider what happens to* $(\alpha + \gamma, 1 - \alpha - \gamma)$ *for some arbitrary* γ*.*

Question 5. (*10 points*) Suppose you bought a light bulb. Let X be the random variable corresponding to the number of years that the light bulb lasts. You have two hypotheses for the distribution of X corresponding to how well the bulb was made:

$$H_1 = \text{“}X \text{ is a geometric random variable with parameter } 2/3\text{.”}$$

$$H_2 = \text{“}X \text{ is a geometric random variable with parameter } 1/2\text{.”}$$

Your priors for these hypotheses are $P(H_1) = 3/5$ and $P(H_2) = 2/5$.

5.1 (*2 points*): *Let $\{X = 1\}$ be the event the light bulb lasts for one year. What are the values:*

$$P(X = 1|H_1) = \qquad P(X = 1|H_2) =$$

$$P(X = 3|H_1) = \qquad P(X = 3|H_2) =$$

5.2 (*2 points*): *If event $\{X = 1\}$ is observed, which is the MAP hypothesis? Show your working.*

5.3 (*2 points*): *If event $\{X = 3\}$ is observed, which is the MAP hypothesis? Show your working.*

5.4 (*2 points*): *What is the value of $P(X = i)$ as a function of i based on the priors above?*

5.5 (*2 points*): *What is the value of $E(X)$? To get full marks your answer should be fully simplified.*

Question 6. (*10 points*) Good news! The computer science vending machine has been updated to include some heathy snacks and Mexican favorites. The available options are:

$$\Omega = \{\text{Apple}, \text{Banana}, \text{Chips}, \text{Donut}, \text{Empanada}\}$$

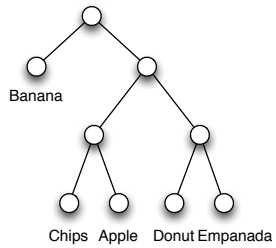
The probabilities of picking these options are:

$P(\text{Apple}) = 4/10$ $P(\text{Banana}) = 2/10$ $P(\text{Chips}) = 2/10$ $P(\text{Donut}) = 1/10$ $P(\text{Empanada}) = 1/10$.

6.1 (*2 points*): *What is the probability of eating a fruit? Hint: An empanada is not a fruit.*

6.2 (*2 points*): *Suppose that to pick one of the items you enter a binary sequence of length k. What is the minimum value of k that can be used such that all items have a different binary sequence.*

In the rest of this question, the sequences are of different lengths and are chosen according to the following tree, e.g., the sequence for "Banana" is 0 and the sequence for "Empanada" is 111.



6.3 (*2 points*): *What are the sequences for:*

$$\begin{aligned} \text{“}\textit{Chips}\text{”} &= \\ \text{“}\textit{Apple}\text{”} &= \\ \text{“}\textit{Donut}\text{”} &= \end{aligned}$$

6.4 (*2 points*): *If I pick an item according to the above probabilities, let X be the length of the corresponding sequence. What is the expected value of X?*

6.5 (*2 points*): **Extra Credit:** *Find a tree in which the expected length of the sequence is smaller. What is the expected length in this case.*

Question 7. (*10 points*) It's the day of the big game and outside the Mullins center there are two competing t-shirt vendors, Alice and Bob. They each paid $100 for a license to sell t-shirts and are each now deciding whether to sell their t-shirts for $10 or $20.

- If both sell their t-shirts at $10 then each will sell 10 t-shirts.
- If both sell their t-shirts at $20 then each will sell 5 t-shirts.
- If Alice sells at $10 and Bob sells at $20 then Alice sells 20 t-shirts and Bob sells none.
- If Bob sells at $10 and Alice sells at $20 then Bob sells 20 t-shirts and Alice sells none.

You may assume that the t-shirts were effectively free to make and so the license is the only cost.

7.1 (*2 points*): *For each combination of strategies, what is Alice's profit? Remember to include the cost of the license and so the profit could be negative.*

$$\begin{aligned} \textit{Alice's profit if both sell at \$10} &= \\ \textit{Alice's profit if both sell at \$20} &= \\ \textit{Alice's profit if Alice sells at \$10 and Bob sells at \$20} &= \\ \textit{Alice's profit if Alice sells at \$20 and Bob sells at \$10} &= \end{aligned}$$

7.2 (*1 points*): *Is this a zero-sum game?*

7.3 (*2 points*): *Suppose Alice sets her price at $10 with probability p. Bob sets his price at $10 with probability q. As a function of p and q, what is the expected profit for Alice?*

7.4 (*1 points*): *What value should Alice choose for p if $q = 1/2$.*

7.5 (*2 points*): *Find a Nash equilibrium for the game. (You just need to find one.)*

7.6 (*2 points*): **Extra Credit:** *Alice is considering bribing Bob to close his t-shirt stand so that she has no competition. Ignoring anti-competition laws and moral issues, what's the maximum she should pay Bob? What assumptions are you making in your answer?*

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

$$\text{For } k \in \{0,1,2,\ldots,N\} : P(X=k) = \binom{N}{k} p^k (1-p)^{N-k} \ , \ E(X) = Np \ , \ var(X) = Np(1-p)$$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

$$\text{For } k \in \{1,2,3,\ldots\} : \ P(X=k) = (1-p)^{k-1} \cdot p \quad , \quad E(X) = \frac{1}{p} \quad , \quad var(X) = (1-p)/p^2$$

- **Poisson Random Variable with parameter** $\lambda > 0$:

$$\text{For } k \in \{0,1,2,\ldots\} : \ P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad E(X) = \lambda \quad , \quad var(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

$$\text{For } k \in \{a, a+1, \ldots b\} : \ P(X=k) = \frac{1}{b-a+1} \ , \ E(X) = \frac{a+b}{2} \ , \ var(X) = \frac{(b-a+1)^2 - 1}{12}$$

Bayes Formula

- If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$