Instructions:

• Answer the questions directly on the exam pages.
• Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
• If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., “9 \times 35! + 2” or “0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)” is fine).
• If you need extra space, use the back of a page.
• No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
• If you have questions during the exam, raise your hand.
• The formulas for some standard random variables can be found on the last page.

<table>
<thead>
<tr>
<th>Question</th>
<th>Value</th>
<th>Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>6 (Extra Credit)</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>55</td>
<td></td>
</tr>
</tbody>
</table>
Question 1. (10 points) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (2 points): If $X$ is a random variable, then the events $\{X = 1\}$ and $\{X = 2\}$ are disjoint.

Answer: True.

1.2 (2 points): If $X$ is a Bernoulli random variable then $E(X) = P(X = 1)$.

Answer: True.

1.3 (2 points): If $X$ is a Binomial random variable with parameters $N = 10$ and $p = 1/2$ then

$$P(X = 1) = P(X = 9).$$

Answer: True.

1.4 (2 points): Two random variables with the same expectation, have the same variance.

Answer: False.

1.5 (2 points): For any random variable $X$, $P(X > E(X)) = P(X < E(X))$.

Answer: False. A counter-example would be $P(X = -4) = 1/4$ and $P(X = 4/3) = 3/4$. Hence, $E(X) = 0$ but $P(X > E(X)) = 3/4 \neq 1/4 = P(X < E(X))$. 


Question 2. * (10 points) * Suppose that \( X \) is a random variable that takes values in the set \{1, 2, 3, 4, 5\} and \( P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = 1/5 \).

2.1 (2 points): * What is the value of \( E(X) \)?

**Answer:** \( E(X) = 1 \times 1/5 + 2 \times 1/5 + 3 \times 1/5 + 4 \times 1/5 + 5 \times 1/5 = 3 \).

2.2 (2 points): * What is the value of \( E(X^2) \)?

**Answer:** \( E(X^2) = 1^2 \times 1/5 + 2^2 \times 1/5 + 3^2 \times 1/5 + 4^2 \times 1/5 + 5^2 \times 1/5 = 55/5 = 11 \).

2.3 (2 points): * What is the value of \( \text{var}(X) \)?

**Answer:** \( \text{var}(X) = E(X^2) - (E(X))^2 = 11 - 3^2 = 2 \).

2.4 (2 points): * What is the exact value of \( P(X \geq 4) \)?

**Answer:** \( P(X \geq 4) = P(X + 4) + P(X + 5) = 2/5 \).

2.5 (2 points): * What is the exact value of \( P(|X - E(X)| \geq 2) \)?

**Answer:** \( P(|X - E(X)| \geq 2) = P(X = 1) + P(X = 5) = 2/5 \).
Question 3. (11 points) During a lecture, an instructor picks a student at random and asks if they know a certain definition from the course. Define the following two random variables $X$ and $Y$. $X$ takes the value 0 or 1 where $X = 1$ if the student knows the answer and $X = 0$ otherwise. $Y$ is the number of the student’s discussion section, i.e., $Y$ is either 1, 2, or 3. The joint probabilities are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
<th>$Y = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$X = 1$</td>
<td>0.05</td>
<td>0.05</td>
<td>0.3</td>
</tr>
</tbody>
</table>

So, for example $P(X = 0, Y = 1) = 0.2$ and $P(X = 1, Y = 1) = 0.05$.

3.1 (1 points): What’s the value of $P(X = 1, Y = 3)$?

3.2 (2 points): Enter the values for the following probabilities:

$P(X = 0) = 0.6$ , $P(X = 1) = 0.4$ , $P(Y = 1) = 0.25$ , $P(Y = 2) = 0.25$ , $P(Y = 3) = 0.5$

3.3 (2 points): What is the value of $P(X = 0|Y = 1)$ and $P(X = 1|Y = 1)$?

Answer: $P(X = 0|Y = 1) = P(X = 0, Y = 1)/P(Y = 1) = 0.2/0.25 = 0.8$ and $P(X = 1|Y = 1) = P(X = 1, Y = 1)/P(Y = 1) = 0.05/0.25 = 0.2$.

3.4 (2 points): Are $X$ and $Y$ independent? Justify your answer.

Answer: No. If they were independent then $P(X = 0, Y = 1) = 0.2$ should equal $P(X = 0)P(Y = 1) = 0.6 \times 0.25 = 0.15$ which isn’t true.

3.5 (2 points): What’s the value of the expectation of $X$ and the expectation of $Y$?

Answer: $E(X) = 0 \times 0.6 + 1 \times 0.4 = 0.4$ and $E(Y) = 1 \times 0.25 + 2 \times 0.25 + 3 \times 0.5 = 2.25$.

3.6 (2 points): What’s the value of the expectation of $XY$ and $cov(X,Y) = E(XY) − E(X)E(Y)$?

Answer: $E(XY) = (0 \times 1) \times 0.2 + (0 \times 2) \times 0.2 + (0 \times 3) \times 0.2 + (1 \times 1) \times 0.05 + (1 \times 2) \times 0.05 + (1 \times 3) \times 0.3 = 0.15 + 0.9 = 1.05$. Therefore, $cov(X,Y) = 1.05 - 0.4 \times 2.25 = 1.05 - 0.9 = 0.15$. 

4
**Question 4.** (12 points) Consider an experiment where we toss a fair coin 100 times. Let $X$ be the number of tails that are observed.

4.1 (3 points): What is the expected value, variance, and standard deviation of $X$?

\[
E(X) = Np = 100 \times 0.5 = 50 \quad \text{var}(X) = Np(1-p) = 100 \times 0.5 \times 0.5 = 25 \quad \sigma(X) = \sqrt{\text{var}(X)} = 5
\]

4.2 (2 points): Use the Chebyshev bound to find an upper bound on the value of $P(|X - 50| \geq 10)$.

**Answer:**

\[
P(|X - 50| \geq 10) = P(|X - E(X)| \geq 10) \leq \frac{\text{var}(X)}{10^2} = \frac{25}{100} = 1/4
\]

4.3 (2 points): Use the Markov bound to find an upper bound on the value of $P(X \geq 60)$.

**Answer:**

\[
P(X \geq 60) \leq \frac{E(X)}{60} = \frac{50}{60} = 5/6.
\]

4.4 (3 points): Let $Y$ be the number of heads that are observed and let $Z = X + Y$. What is the expected value and variance of $Y$ and $Z$?

\[
E(Y) = 100 \times 0.5 = 50 \quad \text{var}(Y) = 100 \times 0.5 \times 0.5 = 25
\]

\[
E(Z) = 50 + 50 \quad \text{by linearity of expectation} \quad \text{var}(Z) = 0 \quad \text{because } Z \text{ is constant.}
\]

4.5 (2 points): Are $Y$ and $Z$ independent? Justify your answer.

**Answer:** $Y$ and $Z$ are independent because $Z$ is constant.
Question 5. (12 points) The students of CMPSCI 245 “Reasoning about Certainty” are taking a midterm. The midterm lasts for a maximum of 120 minutes but if we pick a random student, the expected time taken by the student is only 60 minutes. Let $T$ be the time taken by the student.

5.1 (2 points): What’s the best upper bound you can show for the value of $P(T \geq 90)$?

**Answer:** $P(T \geq 90) \leq E(X)/90 = 60/90 = 2/3$.

5.2 (2 points): What’s the best lower bound you can show for the value of $P(T < 90)$?

**Answer:** $P(T < 90) = 1 - P(T \geq 90) \geq 1 - 2/3 = 1/3$.

5.3 (2 points): What’s the largest value $\text{var}(T)$ can take and what’s the corresponding PMF?

**Answer:** Let $P(T = 0) = 1/2$ and $P(T = 120) = 1/2$, then $\text{var}(T) = 60^2 = 3600$.

5.4 (2 points): What’s the smallest value $\text{var}(T)$ can take and what’s the corresponding PMF?

**Answer:** Let $P(T = 60) = 1$ then $\text{var}(T) = 0$.

5.5 (1 points): If there are 50 students taking CMPSCI 245 and the average time taken for the exam is 60 minutes, what’s the maximum number of students who took the full 120 minutes?

**Answer:** Let $x_1, x_2, \ldots, x_{50}$ be the times taken by the students. Let $t$ be the number of students who took the full 120 minutes. Hence, $60 = \frac{x_1 + x_2 + x_3 + \ldots + x_{50}}{50} \leq \frac{120t}{50}$ and therefore $t \leq 25$.

5.6 (3 points): What’s the best upper bound you can show for the value of $P(T \leq 10)$? **Hint:** Consider a new random variable $R = 120 - T$.

**Answer:** Note that $R$ is non-negative and $E(R) = 120 - E(T) = 60$. Hence, $P(T \leq 10) = P(R \geq 110) \leq 60/110 = 6/11$. 


Question 6.  (5 points) Extra Credit: Suppose $X$ is a geometric random variable with parameter $p$. Hint: You may want to use the formula $1 + x + x^2 + \ldots = 1/(1 - x)$ for $0 < x < 1$.

6.1 (2 points): What’s the probability that $X$ takes an odd value?

**Answer:**

$$P(X \in \{1, 3, 5, 7, \ldots\}) = p + (1-p)^2p + (1-p)^4p + (1-p)^6p + \ldots$$
$$= p(1 + (1-p)^2 + (1-p)^4 + (1-p)^6 + \ldots)$$
$$= \frac{1}{1-(1-p)^2} = \frac{1}{2-p}$$

6.2 (3 points): Let $p = 1/2$. Find a function $f : \mathbb{R} \rightarrow \{0, 1\}$ such that $Y = f(X)$ satisfies $E(Y) = 1/7$. Prove your result.

**Answer:** Let $f(x) = 1$ if $x = 3, 6, 9, 12, \ldots$ and $f(x) = 0$ otherwise. Then

$$P(f(X) = 1) = P(X \in \{3, 6, 9, 12, \ldots\})$$
$$= (1-p)^2p + (1-p)^5p + (1-p)^8p + \ldots$$
$$= 1/8 + 1/8^2 + 1/8^3 + \ldots$$
$$= (1 + 1/8 + 1/8^2 + 1/8^3 + \ldots)/8$$
$$= \frac{1}{8-1} = 1/7$$

and so $E(f(X)) = P(f(X) = 1) = 1/7$ as required.
Standard Random Variables

- Bernoulli Random Variable with parameter \( p \in [0, 1] \):
  \[
P(X = k) = \begin{cases} 
  1 - p & \text{if } k = 0 \\
  p & \text{if } k = 1 
\end{cases}, \quad E(X) = p \quad , \quad \text{var}(X) = p(1 - p)
\]

- Binomial Random Variable with parameters \( p \in [0, 1] \) and \( N \in \{1, 2, 3, \ldots\} \):
  For \( k \in \{0, 1, 2, \ldots, N\} \) : \( P(X = k) = \binom{N}{k} p^k (1-p)^{N-k} \), \( E(X) = Np \), \( \text{var}(X) = Np(1-p) \)

- Geometric Random Variable with parameter \( p \in [0, 1] \):
  For \( k \in \{1, 2, 3, \ldots\} \) : \( P(X = k) = (1 - p)^{k-1} \cdot p \), \( E(X) = \frac{1}{p} \), \( \text{var}(X) = (1 - p)/p^2 \)

- Poisson Random Variable with parameter \( \lambda > 0 \):
  For \( k \in \{0, 1, 2, \ldots\} \) : \( P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \), \( E(X) = \lambda \), \( \text{var}(X) = \lambda \)

- Discrete Uniform Random Variable with parameters \( a, b \in \mathbb{Z} \) and \( a < b \):
  For \( k \in \{a, a+1, \ldots b\} \) : \( P(X = k) = \frac{1}{b-a+1} \), \( E(X) = \frac{a+b}{2} \), \( \text{var}(X) = \frac{(b - a + 1)^2 - 1}{12} \)