CMPSCI 240: Reasoning Under Uncertainty
First Midterm Exam

February 13, 2013. Start 7:00pm. End 9:00pm.

Name: ________________________________ ID: ____________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine). "Choose" notation must be expanded in terms of arithmetic operations in final answers to receive full points.
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6 (Extra Credit)	10	
Total	50	

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *For any two events A and B,*

$$(A \cup B)^c = A^c \cap B^c$$

Answer: True

1.2 (*2 points*): *For any three events A, B, and C where A and B are disjoint and $0 < P(C) < 1$,*

$$P(A \cup B|C) = P(A|C) + P(B|C)$$

Answer: True

1.3 (*2 points*): *For any two events A and B where $0 < P(B) < 1$,*

$$P(A|B) = 1 - P(A|B^c)$$

Answer: False

1.4 (*2 points*): *For any three events A, B, and C,*

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Answer: False

1.5 (*2 points*): *For any two events A and B,*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Answer: True

Question 2. (*10 points*) Suppose you throw a fair 12-sided dice to get a value from the set

$$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \ .$$

Consider the events

$$A = \{1, 2, 3, 4, 11, 12\} \quad , \quad B = \{3, 4, 5, 6, 11, 12\} \quad , \quad C = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

2.1 (*1 points*): *What is the value of $P(A)$?*

Answer: $P(A) = 6/12 = 1/2$

2.2 (*1 points*): *What is the value of $P(B)$?*

Answer: $P(B) = 6/12 = 1/2$

2.3 (*2 points*): *What is the value of $P(A \cap B)$?*

Answer: $P(A \cap B) = |A \cap B|/|\Omega| = 4/12 = 1/3.$

2.4 (*1 points*): *Are the events A and B independent?*

Answer: No, since $P(A \cap B) \neq P(A) \times P(B)$.

2.5 (*1 points*): *What is the value of $P(A|C)$?*

Answer: $P(A|C) = |A \cap C|/|C| = 4/8 = 1/2$

2.6 (*1 points*): *What is the value of $P(B|C)$?*

Answer: $P(B|C) = |B \cap C|/|C| = 4/8 = 1/2$

2.7 (*2 points*): *What is the value of $P(A \cap B|C)$?*

Answer: $P(A \cap B|C) = |A \cap B \cap C|/|C| = 2/8 = 1/4$

2.8 (*1 points*): *Are the events A and B independent conditioned on C?*

Answer: Yes, since $P(A \cap B|C) = P(A|C) \times P(B|C)$.

Question 3. (*10 points*) Suppose you perform an experiment where the sample space is

$$\Omega = \{o_1, o_2, o_3, o_4, o_5, o_6\}$$

and the probability rule satisfies:

$$P(\{o_1\}) = 1/2 \quad , \quad P(\{o_2\}) = 1/4 \quad , \quad P(\{o_3\}) = 1/8 \quad , \quad P(\{o_4\}) = 1/16 \quad , \quad P(\{o_5\}) = 1/32$$

Define the events $A = \{o_1, o_2\}$, $B = \{o_2, o_3\}$, and $C = \{o_1, o_2, o_3, o_4, o_5\}$.

3.1 (*2 points*): *What is the value of* $P(\{o_6\})$*?*

Answer: $P(\{o_6\}) = 1 - 1/2 - 1/4 - 1/8 - 1/16 - 1/32 = 1/32$.

3.2 (*2 points*): *What is the value of* $P(A)$*?*

Answer: $P(A) = P(\{o_1, o_2\}) = 1/2 + 1/4 = 3/4$.

3.3 (*2 points*): *What is the value of* $P(A \cup B)$*?*

Answer: $P(A \cup B) = P(\{o_1, o_2, o_3\}) = 1/2 + 1/4 + 1/8 = 7/8$.

3.4 (*2 points*): *What is the value of* $P(A \cap B)$*?*

Answer: $P(A \cap B) = P(\{o_2\}) = 1/4$.

3.5 (*2 points*): *What is the value of* $P(A \cap B|C)$*?*

Answer: $P(A \cap B|C) = P(A \cap B \cap C)/P(C) = (1/4)/(1 - 1/32) = 8/31$.

Question 4. (*10 points*) Every Tuesday night I go to one of three restaurants: Amherst Chinese, Freshside, or Paradise of India. I go to Amherst Chinese with probability 0.3. I go to Freshside with probability 0.6. I go to Paradise of India with probability 0.1. If I go to Amherst Chinese, I'll eat rice with probability 1. If I go to Freshside, I'll eat rice with probability 0.3. If I go to Paradise of India, I'll eat rice with probability 0.4. Define the events:

$$C = \text{"go to Amherst Chinese"} \quad , \quad F = \text{"go to Freshside"}$$

$$I = \text{"go Paradise of India"} \quad , \quad R = \text{"eat rice"}$$

4.1 (*2 points*): *Enter the values for the following probabilities:*

$P(C) =$ 0.3 , $P(F) =$ 0.6 , $P(I) =$ 0.1 , $P(R|C) =$ 1 , $P(R|F) =$ 0.3 , $P(R|I) =$ 0.4

4.2 (*2 points*): *What is the probability that I'll go to Freshside and eat rice?*

Answer: $P(F \cap R) = P(F)P(R|F) = 0.6 \times 0.3 = 0.18$

4.3 (*2 points*): *What is the probability that I'll go to Paradise of India and not eat rice?*

Answer: $P(I \cap R^c) = P(I)P(R^c|I) = 0.1 \times (1 - 0.4) = 0.06$

4.4 (*2 points*): *What is the probability that I'll eat rice?*

Answer: $P(R) = P(C)P(R|C) + P(F)P(R|F) + P(I)P(R|I) = 0.3 \times 1 + 0.6 \times 0.3 + 0.1 \times 0.4 = 0.52$.

4.5 (*2 points*): *If you know I ate rice, what is the probability that I went to Amherst Chinese?*

Answer: $P(C|R) = P(C \cap R)/P(R) = P(C)P(R|C)/P(R) = 0.3 \times 1/0.52 = 0.577$.

Question 5. (*10 points*) There are 97 students in the class: 36 are in the first discussion section, 37 are in the second discussion section, and 24 are in the third discussion section. Suppose I write the name of each student on a piece of paper and place all the pieces in a hat. I then randomly pick 2 names out of the hat (without replacement).

5.1 (*3 points*): *What's the probability they are both in the first discussion section?*

Answer: Let A be the event that the first student is from the first discussion section and let B be the event that the second student is from the first discussion section. Then

$$P(A \cap B) = P(A)P(B|A) = (36/97) \times (35/96) \ .$$

5.2 (*3 points*): *What's the probability they are both in the same discussion section?*

Answer: Let A_i be the event that the first student is from discussion section i and let B_i be the event that the second student is from discussion section i. Then

$$\begin{aligned} & P((A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3)) \\ = \ & P(A_1 \cap B_1) + P(A_2 \cap B_2) + P(A_3 \cap B_3) \\ = \ & (36/97) \times (35/96) + (37/97) \times (36/96) + (24/97) \times (23/96) \end{aligned}$$

5.3 (*2 points*): *What's the probability they are in different discussion sections?*

Answer: Let A_i be the event that the first student is from the discussion section i and let B_i be the event that the second student is from discussion section i. Then

$$\begin{aligned} & P(((A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3))^c) \\ = \ & 1 - P((A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3)) \\ = \ & 1 - (36/97) \times (35/96) - (37/97) \times (36/96) - (24/97) \times (23/96) \end{aligned}$$

5.4 (*2 points*): *If I keep on picking names out of the hat, what's the probability that the last name is from the third discussion section?*

Answer: 24/97 since the last name is equally likely to be any of the 97 names. And 24 of those names are in the third discussion section.

Question 6. (*10 points*) ***Extra Credit:*** In the Spring 2013 offering of CMPSCI 245 "Reasoning about Certainty" there are six students: Amit, Bob, Charlie, Diane, Ely, and Fiona. Amit, Bob, and Charlie are juniors and Diane, Ely, and Fiona are seniors. In each lecture, some subset of the students are present. Let $S = \{\text{Amit, Bob, Charlie, Diane, Ely, Fiona}\}$. Hint: You'll be able to solve parts 2, 3, 4, and 5 by enumerating all the possibilities (and will get full credit if you do it correctly) but there are more efficient ways.

6.1 (*2 points*): *How many subsets of S are there? Remember to include the empty set and S.*

Answer: $2^6 = 64$.

6.2 (*2 points*): *How many subsets are there with exactly one junior?*

Answer: There are 3 ways to pick which junior and 2^3 ways to pick the subset of seniors so the answer is $3 \times 2^3 = 24$.

6.3 (*2 points*): *How many subsets are there with exactly three students?*

Answer: $\binom{6}{3} = \frac{6\times5\times4}{3\times2\times1} = 20$.

6.4 (*2 points*): *How many subsets are there with the same number of juniors and seniors?*

Answer: There are $\binom{3}{i}$ ways to pick a subset of i juniors and $\binom{3}{i}$ ways to pick a subset of i seniors. Hence, there are $\binom{3}{i} \times \binom{3}{i}$ ways to pick exactly i juniors and i seniors. Therefore the total number of ways of picking the same number of juniors as seniors is:

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 1 + 9 + 9 + 1 = 20$$

6.5 (*2 points*): *How many subsets are there where there are strictly more seniors than juniors?*

Answer: There are $64 - 20 = 44$ ways for there to be a different number of juniors and seniors. In half of these, by symmetry, there are strictly more seniors than juniors, i.e., $44/2 = 22$.