Name: ________________________________  ID: __________________________

Instructions:

• Answer the questions directly on the exam pages.

• Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.

• If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., “9 × 35! + 2” or “0.5 × 0.3/(0.2 × 0.5 + 0.9 × 0.1)” is fine). “Choose” notation must be expanded in terms of arithmetic operations in final answers to receive full points.

• If you need extra space, use the back of a page.

• No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.

• If you have questions during the exam, raise your hand.

<table>
<thead>
<tr>
<th>Question</th>
<th>Value</th>
<th>Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6 (Extra Credit)</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>
Question 1. (10 points) Indicate whether each of the following statements is TRUE or FALSE. No justification is required.

1.1 (2 points): For any two events $A$ and $B$,

$$(A \cup B)^c = A^c \cap B^c$$

Answer: True

1.2 (2 points): For any three events $A, B,$ and $C$ where $A$ and $B$ are disjoint and $0 < P(C) < 1$,

$$P(A \cup B|C) = P(A|C) + P(B|C)$$

Answer: True

1.3 (2 points): For any two events $A$ and $B$ where $0 < P(B) < 1$,

$$P(A|B) = 1 - P(A|B^c)$$

Answer: False

1.4 (2 points): For any three events $A, B,$ and $C$,

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Answer: False

1.5 (2 points): For any two events $A$ and $B$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Answer: True
Question 2. (10 points) Suppose you throw a fair 12-sided dice to get a value from the set
\[ \Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \text{.} \]
Consider the events
\[ A = \{1, 2, 3, 4, 11, 12\} \text{,} \quad B = \{3, 4, 5, 6, 11, 12\} \text{,} \quad C = \{1, 2, 3, 4, 5, 6, 7, 8\} \]

2.1 (1 points): What is the value of \( P(A) \)?
Answer: \( P(A) = \frac{6}{12} = \frac{1}{2} \)

2.2 (1 points): What is the value of \( P(B) \)?
Answer: \( P(B) = \frac{6}{12} = \frac{1}{2} \)

2.3 (2 points): What is the value of \( P(A \cap B) \)?
Answer: \( P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{4}{12} = \frac{1}{3} \).

2.4 (1 points): Are the events \( A \) and \( B \) independent?
Answer: No, since \( P(A \cap B) \neq P(A) \times P(B) \).

2.5 (1 points): What is the value of \( P(A|C) \)?
Answer: \( P(A|C) = \frac{|A \cap C|}{|C|} = \frac{4}{8} = \frac{1}{2} \)

2.6 (1 points): What is the value of \( P(B|C) \)?
Answer: \( P(B|C) = \frac{|B \cap C|}{|C|} = \frac{4}{8} = \frac{1}{2} \)

2.7 (2 points): What is the value of \( P(A \cap B|C) \)?
Answer: \( P(A \cap B|C) = \frac{|A \cap B \cap C|}{|C|} = \frac{2}{8} = \frac{1}{4} \)

2.8 (1 points): Are the events \( A \) and \( B \) independent conditioned on \( C \)?
Answer: Yes, since \( P(A \cap B|C) = P(A|C) \times P(B|C) \).
Question 3. (10 points) Suppose you perform an experiment where the sample space is

$$\Omega = \{o_1, o_2, o_3, o_4, o_5, o_6\}$$

and the probability rule satisfies:

$$P(\{o_1\}) = \frac{1}{2}, \quad P(\{o_2\}) = \frac{1}{4}, \quad P(\{o_3\}) = \frac{1}{8}, \quad P(\{o_4\}) = \frac{1}{16}, \quad P(\{o_5\}) = \frac{1}{32}$$

Define the events

$$A = \{o_1, o_2\}, \quad B = \{o_2, o_3\}, \quad C = \{o_1, o_2, o_3, o_4, o_5\}.$$ 

3.1 (2 points): What is the value of $$P(\{o_6\})$$?

**Answer:**

$$P(\{o_6\}) = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} = \frac{1}{32}.$$ 

3.2 (2 points): What is the value of $$P(A)$$?

**Answer:**

$$P(A) = P(\{o_1, o_2\}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$ 

3.3 (2 points): What is the value of $$P(A \cup B)$$?

**Answer:**

$$P(A \cup B) = P(\{o_1, o_2, o_3\}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$ 

3.4 (2 points): What is the value of $$P(A \cap B)$$?

**Answer:**

$$P(A \cap B) = P(\{o_2\}) = \frac{1}{4}.$$ 

3.5 (2 points): What is the value of $$P(A \cap B | C)$$?

**Answer:**

$$P(A \cap B | C) = P(A \cap B \cap C) / P(C) = (\frac{1}{4}) / (1 - \frac{1}{32}) = \frac{8}{31}.$$
Question 4. (10 points) Every Tuesday night I go to one of three restaurants: Amherst Chinese, Freshside, or Paradise of India. I go to Amherst Chinese with probability 0.3. I go to Freshside with probability 0.6. I go to Paradise of India with probability 0.1. If I go to Amherst Chinese, I’ll eat rice with probability 1. If I go to Freshside, I’ll eat rice with probability 0.3. If I go to Paradise of India, I’ll eat rice with probability 0.4. Define the events:

\[ C = \text{“go to Amherst Chinese”}, \quad F = \text{“go to Freshside”} \]
\[ I = \text{“go Paradise of India”}, \quad R = \text{“eat rice”} \]

4.1 (2 points): Enter the values for the following probabilities:
\[ P(C) = 0.3, \quad P(F) = 0.6, \quad P(I) = 0.1, \quad P(R|C) = 1, \quad p(R|F) = 0.3, \quad P(R|I) = 0.4 \]

4.2 (2 points): What is the probability that I’ll go to Freshside and eat rice?
Answer: \[ P(F \cap R) = P(F)P(R|F) = 0.6 \times 0.3 = 0.18 \]

4.3 (2 points): What is the probability that I’ll go to Paradise of India and not eat rice?
Answer: \[ P(I \cap R^c) = P(I)P(R^c|I) = 0.1 \times (1 - 0.4) = 0.06 \]

4.4 (2 points): What is the probability that I’ll eat rice?
Answer: \[ P(R) = P(C)P(R|C) + P(F)P(R|F) + P(I)P(R|I) = 0.3 \times 1 + 0.6 \times 0.3 + 0.1 \times 0.4 = 0.52. \]

4.5 (2 points): If you know I ate rice, what is the probability that I went to Amherst Chinese?
Answer: \[ P(C|R) = P(C \cap R) / P(R) = P(C)P(R|C) / P(R) = 0.3 \times 1 / 0.52 = 0.577. \]
Question 5. (10 points) There are 97 students in the class: 36 are in the first discussion section, 37 are in the second discussion section, and 24 are in the third discussion section. Suppose I write the name of each student on a piece of paper and place all the pieces in a hat. I then randomly pick 2 names out of the hat (without replacement).

5.1 (3 points): What’s the probability they are both in the first discussion section?

**Answer:** Let $A$ be the event that the first student is from the first discussion section and let $B$ be the event that the second student is from the first discussion section. Then

$$P(A \cap B) = P(A)P(B|A) = \frac{36}{97} \times \frac{35}{96}.$$ 

5.2 (3 points): What’s the probability they are both in the same discussion section?

**Answer:** Let $A_i$ be the event that the first student is from discussion section $i$ and let $B_i$ be the event that the second student is from discussion section $i$. Then

$$P((A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3)) = P(A_1 \cap B_1) + P(A_2 \cap B_2) + P(A_3 \cap B_3) = \frac{36}{97} \times \frac{35}{96} + \frac{37}{97} \times \frac{36}{96} + \frac{24}{97} \times \frac{23}{96}.$$ 

5.3 (2 points): What’s the probability they are in different discussion sections?

**Answer:** Let $A_i$ be the event that the first student is from the discussion section $i$ and let $B_i$ be the event that the second student is from discussion section $i$. Then

$$P((A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3))^c) = 1 - P((A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3)) = 1 - \frac{36}{97} \times \frac{35}{96} - \frac{37}{97} \times \frac{36}{96} - \frac{24}{97} \times \frac{23}{96}.$$ 

5.4 (2 points): If I keep on picking names out of the hat, what’s the probability that the last name is from the third discussion section?

**Answer:** 24/97 since the last name is equally likely to be any of the 97 names. And 24 of those names are in the third discussion section.
Question 6. (10 points) Extra Credit: In the Spring 2013 offering of CMPSCI 245 “Reasoning about Certainty” there are six students: Amit, Bob, Charlie, Diane, Ely, and Fiona. Amit, Bob, and Charlie are juniors and Diane, Ely, and Fiona are seniors. In each lecture, some subset of the students are present. Let $S = \{\text{Amit, Bob, Charlie, Diane, Ely, Fiona}\}$. Hint: You’ll be able to solve parts 2, 3, 4, and 5 by enumerating all the possibilities (and will get full credit if you do it correctly) but there are more efficient ways.

6.1 (2 points): How many subsets of $S$ are there? Remember to include the empty set and $S$.

Answer: $2^6 = 64$.

6.2 (2 points): How many subsets are there with exactly one junior?

Answer: There are $3$ ways to pick which junior and $2^3$ ways to pick the subset of seniors so the answer is $3 \times 2^3 = 24$.

6.3 (2 points): How many subsets are there with exactly three students?

Answer: $(\binom{6}{3}) = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$.

6.4 (2 points): How many subsets are there with the same number of juniors and seniors?

Answer: There are $\binom{3}{i}$ ways to pick a subset of $i$ juniors and $\binom{3}{i}$ ways to pick a subset of $i$ seniors. Hence, there are $\binom{3}{i} \times \binom{3}{i}$ ways to pick exactly $i$ juniors and $i$ seniors. Therefore the total number of ways of picking the same number of juniors as seniors is:

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 1 + 9 + 9 + 1 = 20$$

6.5 (2 points): How many subsets are there where there are strictly more seniors than juniors?

Answer: There are $64 - 20 = 44$ ways for there to be a different number of juniors and seniors. In half of these, by symmetry, there are strictly more seniors than juniors, i.e., $44/2 = 22$. 

7