CMPSCI 240: Reasoning Under Uncertainty
Final Exam

May 7, 2013.

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Giving more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, you may give your answer using arithmetic operations, such as addition, multiplication, and factorial (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine) unless the problem says otherwise.
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.
- The formulas for some standard random variables can be found on the last page.

Question	Value	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7 (Extra Credit)	5	
Total	60+5	

Question 1. (*10 points*) Indicate whether each of the following five statements is TRUE or FALSE. No justification is required.

1.1 (*2 points*): *If A and B are disjoint events then $P(A \cap B) = 0$.*

1.2 (*2 points*): *If A and B are independent events then A and B are also independent conditioned on any other event C.*

1.3 (*2 points*): *Every two-player game where both players have a finite number of pure strategies has a Nash equilibrium.*

1.4 (*2 points*): *For any random variable X and $t > 0$, $P(X \geq E(X) + t) \leq var(X)/t^2$.*

1.5 (*2 points*): *If X is a geometric random variable with parameter p, then $P(X \geq t) = (1-p)^{t-1}$.*

Question 2. (*10 points*) Let X be a random variable where $P(X = 1) = 1/2, P(X = 2) = 1/4$, and $P(X = 4) = 1/4$. Let $Y = 4X + 1$ be another random variable that depends on X. To get full marks you need to simplify your answers to a single number.

2.1 (*3 points*): *Compute the following quantities:*

$$E(X) = \qquad E(X^2) = \qquad var(X) =$$

2.2 (*2 points*): *Compute the following quantities:*

$$E(Y) = \qquad var(Y) =$$

2.3 (*3 points*): *Compute the following quantities:*

$$E(X + Y) = \qquad var(X + Y) = \qquad \sigma(X + Y) =$$

2.4 (*2 points*): *Recall that the covariance of X and Y is defined as $cov(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$. Compute the covariance of X and Y.*

$$cov(X, Y) =$$

Question 3. (*10 points*) Consider the Markov chain with three states $\{s_1, s_2, s_3\}$ and transition matrix M:

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \end{pmatrix}$$

For example, if you are in state s_3 then the probability you would move to state s_2 is 1.

3.1 (*2 points*): *Draw the transition diagram of the Markov chain and label each edge with the appropriate transition probability.*

3.2 (*2 points*): *Circle the* two *following terms which apply to the Markov chain.*

irreducible reducible aperiodic periodic

3.3 (*2 points*): *If the chain is in state s_1, what's the probability it's in state s_1 after 2 steps?*

3.4 (*1 points*): *If the chain is in state s_1, what's the probability it's in state s_1 after 3 steps?*

3.5 (*3 points*): *What's the steady state distribution of the Markov chain?*

Question 4. (*10 points*) Your magician friend gives you a coin. You have two hypotheses:

H_1 = "it is a normal coin with equal chance of landing heads or tails."

H_2 = "it is a special coin where both sides are tails."

Your prior probabilities for these hypotheses are $P(H_1) = 8/9$ and $P(H_2) = 1/9$. Let D be the event that when you toss the coin two times it lands tails twice.

4.1 (*2 points*): *What's the value of the likelihoods:*

$$P(D|H_1) = \qquad\qquad P(D|H_2) =$$

4.2 (*2 points*): *Given that event D is observed, which is the maximum a posteri (MAP) hypothesis? Show your working.*

4.3 (*2 points*): *What is the value of $P(D)$ based on the priors above?*

4.4 (*2 points*): *Let F be the event that your friend tells you the coin is special. You don't fully trust your friend and assume the following likelihoods $P(F|H_1) = 1/3$ and $P(F|H_2) = 1$. If events D and F are both observed which hypothesis should you pick? You may assume that your friend talks to you before you start tossing the coin.*

4.5 (*2 points*): *Let $X = i$ if the ith hypothesis is true. Let $Y = 1$ if event D occurs and $Y = 0$ otherwise. Let $Z = 1$ if event F occurs and $Z = 0$ otherwise. Draw the Bayesian network for X, Y, and Z based on the scenario above.*

Question 5. (*10 points*) Suppose you store a 128 bit file, i.e., a binary string of length 128, on a really old hard drive that can support such small files. Because the hard drive is unreliable, the file might become corrupted. In particular, each bit is independently flipped (i.e., a 0 changes to a 1 or a 1 changes to a 0) with probability 1/128. Let X be the number of bits that are flipped.

5.1 (*3 points*): *What is the value of the following quantities:*

$P(X = 0) =$ $E(X) =$ $var(X) =$

5.2 (*2 points*): *Let A be the event that the first bit was flipped. What are the following probabilities?*

$P(X = 2) =$ $P(A|X = 2) =$

5.3 (*2 points*): *Suppose the file format for 128 bit files has some built-in error detection. In particular, a file is in the format if and only if the number of ones is even. Write an expression for the probability that a file with this property still satisfies the property after the bits are flipped.*

5.4 (*2 points*): *Because the hard drive is unreliable, you decide to store 5 separate copies of the file. What is the probability that the majority (i.e., 3 or more) of the copies do not become corrupted? You may assume that the bits in the copies are independently flipped with probability* 1/128.

5.5 (*1 points*): *When you need to retrieve the file, what would you do if all 5 copies are no longer identical? In particular, how would you take into account the file format mentioned above?*

Question 6. (*10 points*) Suppose the towns of Amherst and Northampton go to war. They move their troops to either side of the Connecticut River and the situation is very tense.

- If both sides attack, both towns lose 10 points.
- If both sides withdraw, both towns win 0 points.
- If one side attacks and the other side withdraws, the attacking side wins 1 point and the withdrawing side loses 1 point.

Suppose Amherst attacks with probability p and withdraws with probability $1 - p$. Northampton attacks with probability q and withdraws with probability $1 - q$.

6.1 (*2 points*): *What is the pay-off matrix for this game?*

6.2 (*2 points*): *As a function of p and q, what is the expected payoff for Amherst?*

6.3 (*1 points*): *If Amherst Intelligence finds out Northampton will definitely withdraw, i.e., $q = 0$, what strategy should Amherst adopt?*

6.4 (*2 points*): *If Amherst Intelligence finds out $q = 1/2$, what strategy should Amherst adopt and what would be the expected payoff for Amherst?*

6.5 (*3 points*): *Find* all *Nash equilibriums for the game.*

Question 7. (*5 points*) ***Extra Credit:*** Suppose you have a biased coin where the probability of heads is $p > 0$ and the probability of tails is $(1 - p) > 0$. Unfortunately p is unknown.

7.1 (*3 points*): *Consider the following experiment for measuring the bias of the coin: toss the coin t times and let X be the number of times the coin lands heads. For your estimate of p, you return $\tilde{p} = X/t$. Use the Chebyshev bound to estimate $P(|p - \tilde{p}| \geq 1/100)$ and find a value of t such that $P(|p - \tilde{p}| \geq 1/100) \leq 1/100$.*

7.2 (*2 points*): *Design an experiment using your biased coin and define a random variable X based on the outcomes such that $P(X = 0) = 1/2$ and $P(X = 1) = 1/2$.*

Standard Random Variables

- **Bernoulli Random Variable with parameter** $p \in [0,1]$:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases} \quad , \quad E(X) = p \quad , \quad var(X) = p(1-p)$$

- **Binomial Random Variable with parameters** $p \in [0,1]$ **and** $N \in \{1,2,3,\ldots\}$:

$$\text{For } k \in \{0,1,2,\ldots,N\} : P(X=k) = \binom{N}{k} p^k (1-p)^{N-k} \ , \ E(X) = Np \ , \ var(X) = Np(1-p)$$

- **Geometric Random Variable with parameter** $p \in [0,1]$:

$$\text{For } k \in \{1,2,3,\ldots\} : \ P(X=k) = (1-p)^{k-1} \cdot p \quad , \quad E(X) = \frac{1}{p} \quad , \quad var(X) = (1-p)/p^2$$

- **Poisson Random Variable with parameter** $\lambda > 0$:

$$\text{For } k \in \{0,1,2,\ldots\} : \ P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad E(X) = \lambda \quad , \quad var(X) = \lambda$$

- **Discrete Uniform Random Variable with parameters** $a, b \in \mathbb{Z}$ **and** $a < b$:

$$\text{For } k \in \{a, a+1, \ldots b\} : \ P(X=k) = \frac{1}{b-a+1} \ , \ E(X) = \frac{a+b}{2} \ , \ var(X) = \frac{(b-a+1)^2 - 1}{12}$$

Bayes Formula

- If $A_1, \ldots, A_n$ partition Ω then for any event B:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$