H250: Honors Colloquium – Introduction to Computation

Resolution in Predicate Logic

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Review: Resolution in propositional logic

Resolution is an *inference rule* that produces a *new clause* from two clauses with *complementary literals* (p and $\neg p$).

$$\frac{p \vee A \qquad \neg p \vee B}{A \vee B} \quad \textit{resolution}$$

"From clauses $p \vee A$ and $\neg p \vee B$ we derive clause $A \vee B$"

New clause = *resolvent* of the two clauses with respect to p

Example: $res_p(p \vee q \vee \neg r, \neg p \vee s) = q \vee \neg r \vee s$

Resolution is a valid inference rule:
any assignment making premises true also makes conclusion true

$$\{p \vee A, \neg p \vee B\} \models A \vee B$$

Corollary: if $A \vee B$ is a contradiction, so is $(p \vee A) \wedge (\neg p \vee B)$
if resolution reaches contradiction, we started from a contradiction

Resolution For Predicates

In predicate logic, a *literal* is a (possibly negated) predicate:
not p and $\neg p$, but $P(arg1)$ and $\neg P(arg2)$ (different args)

To derive a new clause from $A \vee P(arg1)$ and $B \vee \neg P(arg2)$ must bring args to common form.

Variables in clauses will be (implicitly) universally quantified
can take any value $\Rightarrow$can *substitute* with any *terms*

Is there a substitution bringing the arguments to a common form?
Ex. 1: $P(x, g(y))$ and $P(a, z)$
Ex. 2: $P(x, g(y))$ and $P(z, a)$

Ex. 1: $x \mapsto a$, $z \mapsto g(y)$ yields $P(a, g(y))$, $P(a, g(y)) \Rightarrow$ same

Ex. 2: can't substitute *constant* a with $g(y)$ (a is not a variable)
g is an arbitrary function, don't know if y exists with $g(y) = a$

Substitution and Term Unification

A *substitution* is a *function* that associates *terms* to *variables*

$$\{x_1 \mapsto t_1,\ \ldots, x_n \mapsto t_n\}$$

Two terms can be *unified* if there is a substitution that makes them equal

$f(x, g(y, z), t)\{x \mapsto h(z), y \mapsto h(b), t \mapsto u\} = f(h(z), g(h(b), z), u)$

Unification Rules

A *variable* x may be unified with any *term* t (substitution)
if x *does not occur* in t (otherwise, we'd get an infinite term)
can't unify: x with $f(h(y), g(x, z))$; but can trivially unify x with x

Two *terms* $f(\ldots)$ can be unified if they have the same function f and the *arguments* (terms) can be unified one by one

$\Rightarrow$ two *constants* (0-arg functions): unified if equal

Implementing Unification: Union-Find

Unification defines equivalence classes:
If we unify x with y and then y with $f(z, a)$,
then x is also unified with $f(z, a) \Rightarrow$ *equivalence*

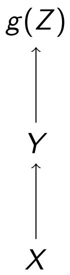
Must track equivalent variables.

Union-Find: data structure for building equivalence classes

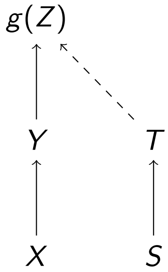
Operations:
find(element): finds representative of equivalence class
union(elem1, elem2): makes elements equivalent (will stay so)

Union-Find Example

One implementation: set of *trees* with links up to parent
  find: returns root of tree
  union: links one root to the other



$find(X) = find(Y) = g(Z)$



$union(Y, S)$
links $find(S)$ with $find(Y)$

We maintain a *map* of variables to *terms*.

Union-Find Example

Unify $f(x, g(x, s(z)), t)$ with $f(h(z), g(h(b), u), z)$

x with $h(z) \Rightarrow \{x \mapsto h(z)\}$

$g(x, s(z))$ with $g(h(b), u) \Rightarrow$

x with $h(b) \Rightarrow h(z)$ with $h(b) \Rightarrow \{x \mapsto h(z), z \mapsto b\}$

$s(z)$ with $u \Rightarrow \{x \mapsto h(z), z \mapsto b, u \mapsto s(z)\}$

t with $z \Rightarrow t$ with $b \Rightarrow \{x \mapsto f(z), z \mapsto b, u \mapsto s(z), t \mapsto b\}$

Substituting all the way:

$\{x \mapsto f(b), z \mapsto b, u \mapsto s(b), t \mapsto b\}$

This substitution is the *most general unifier* of the given terms.

Resolution in Predicate Logic

Take clauses A with $P(...)$ (*positive*) and B, with $\neg P(...)$ (*negated*)

A: $P(x, g(y)) \vee P(h(a), z) \vee Q(z)$ $\quad$ B: $\neg P(h(z), t) \vee R(t, z)$

Choose *some* (≥ 1) $P(...)$ from A and *some* $\neg P(...)$ from B

Rename common variables (A and B are independent clauses)

A: $P(x, g(y)) \vee P(h(a), z) \vee Q(z)$ $\quad$ B: $\neg P(h(z_2), t) \vee R(t, z_2)$

Unify all chosen $P(...)$ from A and $\neg P(...)$ from B

$\{P(x, g(y)), P(h(a), z), P(h(z_2), t)\}$ $\quad$ $x \mapsto h(a); z_2 \mapsto a; z, t \mapsto g(y)$

Eliminate chosen $P(...)$ and $\neg P(...)$ from $A \vee B$.

Apply resulting substitution and *add* new clause to list

$$Q(g(y)) \vee R(g(y), a)$$

Keep original clauses for unification with other predicates

Transforming the Formula for Resolution

We proceed similarly to CNF transformation, but with extra steps.

- rewrite $\rightarrow$, $\leftrightarrow$, etc., keep only $\wedge$, $\vee$,, $\neg$
- push negation inwards to predicates (negation normal form)
- rename to get unique variable names (we'll remove quantifiers)

  $\forall x\, P(x) \vee \forall x\, \exists y\, Q(x, y)$ becomes $\forall x\, P(x) \vee \forall z\, \exists y\, Q(z, y)$

Skolemization: Removing Existential Quantifiers

We want to only keep universally quantified variables, and make quantification implicit.
$\Rightarrow$ use Instantiation for existential quantifiers
In $\forall x_1 ... \forall x_k \exists y : (...)$, the choice of y *depends* on $x_1, \ldots x_k$;
introduce a new *Skolem function* $y = f(x_1, \ldots, x_k)$, eliminating y
i.e., instantiate y with $f(x_1, \ldots, x_k)$

In particular($k = 0$), $\exists y$ outside any $\forall$ is instantiated with a new *Skolem constant*

New function or constant names for *each* existential quantifier.

Transformation Steps (cont'd.)

- Bring all $\forall$ quantifiers to the front (prenex normal form)
  $\forall x\, P(x) \wedge \forall y\, Q(y) \quad \leftrightarrow \quad \forall x \forall y\, (P(x) \wedge Q(y))$
  $\forall x\, P(x) \vee \forall y\, Q(y) \quad \leftrightarrow \quad \forall x \forall y\, (P(x) \vee Q(y))$
- Remove all quantifiers (implicit universal quantification)
- Bring $\wedge$ outside $\vee$ (convert to clausal form)

A Resolution Exercise

`https://www.cs.utexas.edu/users/novak/reso.html`,
Exercise 9:

Every investor bought [something that is] stocks or bonds.
A_1: $\forall X(inv(X) \rightarrow \exists\, Y(buy(X, Y) \wedge (share(Y) \vee bond(Y))))$

If the Dow-Jones Average crashes, then all stocks that are not gold stocks fall.
A_2: $crash \rightarrow \forall\, X(share(X) \wedge \neg gold(X) \rightarrow falls(X))$

If the T-Bill interest rate rises, then all bonds fall.
A_3: $tbrise \rightarrow \forall\, X(bond(X) \rightarrow falls(X))$

Every investor who bought something that falls is not happy.
A_4: $\forall X(inv(X) \rightarrow (\exists\, Y(buy(X, Y) \wedge falls(Y)) \rightarrow \neg happy(X)))$

Resolution Example (cont'd.)

If the Dow-Jones Average crashes and the T-Bill interest rate rises, then any investor who is happy bought some gold stock.

C: $crash \wedge tbrise \rightarrow$
$\forall X(inv(X) \wedge happy(X) \rightarrow \exists Y(buy(X, Y) \wedge share(Y) \wedge gold(Y)))$

We negate the conclusion *before* transforming quantifiers!

$\neg C$: $\neg(crash \wedge tbrise \rightarrow$
$\forall X(inv(X) \wedge happy(X) \rightarrow \exists Y(buy(X, Y) \wedge share(Y) \wedge gold(Y))))$

Example in Negation Normal Form

A_1: $\forall X(\neg inv(X) \vee \exists Y(buy(X,Y) \wedge (share(Y) \vee bond(Y))))$

A_2: $\neg crash \vee \forall X(\neg share(X) \vee gold(X) \vee falls(X))$

A_3: $\neg tbrise \vee \forall X(\neg bond(X) \vee falls(X))$

A_4: $\forall X(\neg inv(X) \vee \forall Y(\neg buy(X,Y) \vee \neg falls(Y)) \vee \neg happy(X))$

$\neg C$: $crash \wedge tbrise \wedge$
$\exists X(inv(X) \wedge happy(X) \wedge \forall Y(\neg buy(X,Y) \vee \neg share(Y) \vee \neg gold(Y)))$

Example: Skolemization

A_1: $\forall X(\neg inv(X) \vee \exists Y(buy(X,Y) \wedge (share(Y) \vee bond(Y))))$
Item Y bought depends on investor X, $Y = f(X)$

$\forall X(\neg inv(X) \vee (buy(X,f(X)) \wedge (share(f(X)) \vee bond(f(X)))))$

X in $\exists X(...)$ becomes constant b
$\neg C$: $crash \wedge tbrise \wedge \exists X(inv(X) \wedge happy(X)$
$\wedge \forall Y(\neg buy(X,Y) \vee \neg share(Y) \vee \neg gold(Y)))$
$crash \wedge tbrise \wedge inv(b) \wedge happy(b)$
$\wedge \forall C(\neg buy(b,Y) \vee \neg share(Y) \vee \neg gold(Y))$

Example: Eliminating Quantifiers

All remaining variables are arbitrary (implicit universal quantification)

A_1: $\neg inv(X) \vee (buy(X, f(X)) \wedge (share(f(X)) \vee bond(f(X))))$

A_2: $\neg crash \vee \neg share(X) \vee gold(X) \vee falls(X)$

A_3: $\neg tbrise \vee \neg bond(X) \vee falls(X)$

A_4: $\neg inv(X) \vee \neg buy(X, Y) \vee \neg falls(Y) \vee \neg happy(X)$

$\neg C$: $crash \wedge tbrise \wedge inv(b) \wedge happy(b)$
$\wedge(\neg buy(b, Y) \vee \neg share(Y) \vee \neg gold(Y))$

Next: apply distributivity of $\vee$ over $\wedge$ and write clauses separately

Example: Clausal Form

(1) $\neg inv(X) \vee buy(X, f(X))$

(2) $\neg inv(X) \vee share(f(X)) \vee bond(f(X)))$

(3) $\neg crash \vee \neg share(X) \vee gold(X) \vee falls(X)$

(4) $\neg tbrise \vee \neg bond(X) \vee falls(X)$

(5) $\neg inv(X) \vee \neg buy(X, Y) \vee \neg falls(Y) \vee \neg happy(X)$

(6) $crash$

(7) $tbrise$

(8) $inv(b)$

(9) $happy(b)$

(10) $\neg buy(b, Y) \vee \neg share(Y) \vee \neg gold(Y)$

Proof: Generating Resolvents

(1) $\neg inv(X) \lor buy(X, f(X))$

(2) $\neg inv(X) \lor share(f(X)) \lor bond(f(X)))$

(3) $\neg crash \lor \neg share(X) \lor gold(X) \lor falls(X)$

(4) $\neg tbrise \lor \neg bond(X) \lor falls(X)$

(5) $\neg inv(X) \lor \neg buy(X, Y) \lor \neg falls(Y) \lor \neg happy(X)$

(6) $crash$

(7) $tbrise$

(8) $inv(b)$

(9) $happy(b)$

(10) $\neg buy(b, Y) \lor \neg share(Y) \lor \neg gold(Y)$

Try to progressively eliminate predicates

(11) $\neg share(X) \lor gold(X) \lor falls(X)$ (3, 6)

(12) $\neg buy(b, Y) \lor \neg share(Y) \lor falls(Y)$ (10, 11, $X = Y$)

(13) $\neg bond(X) \lor falls(X)$ (4, 7)

Deriving Empty Clause

(1) $\neg inv(X) \vee buy(X, f(X))$

(2) $\neg inv(X) \vee share(f(X)) \vee bond(f(X)))$

(5) $\neg inv(X) \vee \neg buy(X, Y) \vee \neg falls(Y) \vee \neg happy(X)$

(8) $inv(b)$

(9) $happy(b)$

(12) $\neg buy(b, Y) \vee \neg share(Y) \vee falls(Y)$ (10, 11, $X = Y$)

(13) $\neg bond(Y) \vee falls(Y)$ rename for unification with (2)

(14) $\neg inv(X) \vee share(f(X)) \vee falls(f(X))$ (2, 13, $Y = X$)

(15) $\neg buy(b, f(X)) \vee \neg inv(X) \vee falls(f(X))$ (12, 14, $Y = f(X)$)

(16) $\neg buy(b, Y) \vee \neg falls(Y) \vee \neg happy(b)$ (5, 8, $X = b$)

(17) $\neg buy(b, Y) \vee \neg falls(Y)$ (9, 16)

(18) $\neg buy(b, f(X)) \vee \neg inv(X)$ (15, 17, $Y = f(X)$)

(19) $\neg inv(b)$ (1, 18, $X = b$)

(20) $\emptyset$ (proof by contradiction done) (8, 19)

Revisiting Resolution

We try to prove:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_n \rightarrow C$$

by contradiction, negating the conclusion and showing

$A_1 \wedge A_2 \wedge \ldots \wedge A_n \wedge \neg C$ is a *contradiction*

We repeatedly generate new clauses (*resolvents*) by resolution with unification.

If we get the *empty clause*, the initial formula is *unsatisfiable*
If we *can't find new resolvents*, the formula is *satisfiable*

Resolution in predicate logic is *refutation-complete*
for any unsatisfiable formula, we'll get the empty clause
but can't determine satisfiability of *any* formula
(there are formulas for which the procedure never stops)