H250: Honors Colloquium – Introduction to Computation

Representing Propositional Formulas

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Colloquium Topics (tentative)

Binary decision diagrams
Conjunctive normal form
SAT checking
Axiomatization of logic
Resolution theorem proving, unification
Fixpoint theorem
Representing and exploring search spaces
Number Theory
Combinatorics
Linear recurrences
Recursive tree traversals
Grammars and parsing
Automata properties and testing
Automata learning

Logistics

Grading:

70% homeworks ($\sim$1 problem / week)

30% final project (discuss/agree topic by Columbus Day)

Q&A: Campuswire

Today

Binary Decision Diagrams

Conjunctive Normal Form

Tseitin Transformation

Review: Propositional Formulas

Boolean operators: $\neg, \wedge, \vee$.
We'll rewrite $\rightarrow, \leftrightarrow, \oplus$ in these terms.

Interesting Questions:

Are two formulas A and B equivalent? $A \leftrightarrow B$

Is a formula a *tautology* ? (true in *all truth assignments*)

Is a formula *satisfiable*? (in *at least one* truth assignment)

Is a formula a *contradiction*? (has *no* satisfying assignment)

Representing Propositional Formulas

Ideally, a representation should be

canonical (a formula is represented in a single way)
two formulas are equal precisely if they have same representation

simple and *compact* (easy to implement and store)

easy to process (simple, efficient algorithms)

Binary decision diagrams are such a representation for Boolean formulas (Bryant, 1986)

Review: Truth tables

Truth tables show the value of a formula for all truth assignments (all interpretations).

Two formulas are *equivalent* if they have the *same truth table*

But: 2^n combinations (lines) for a formula with n propositions

a	b	c	$a \rightarrow (b \rightarrow c)$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

a	b	c	$(a \rightarrow b) \rightarrow c$
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

Boole expansion / Shannon decomposition wrt a variable

By fixing the value of a variable, the formula simplifies

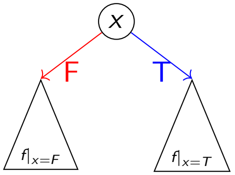
Let $f = (a \vee b) \wedge (a \vee c) \wedge (\neg a \vee \neg b \vee c)$.
We assign values T and F to variable a (two halves of truth table)

$f|_{a=T} = T \wedge T \wedge (\neg b \vee c) = \neg b \vee c$ $\qquad$ $f|_{a=F} = b \wedge c \wedge T = b \wedge c$

Boole expansion
(or *Shannon decomposition*)

$$\boxed{f = x \wedge f|_{x=T} \vee \neg x \wedge f|_{x=F}}$$

expresses a Boolean function f
with respect to a variable x



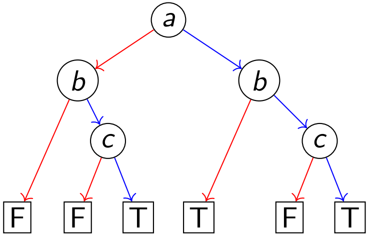
In code (ML): if-then-else with variable as condition

```
if a then not b || c else b && c
```

Binary decision tree

Continuing with subformulas, we obtain a *decision tree*:
assigning variables and *following branches* (true/false),
we obtain the function value (T/F, or 0/1)

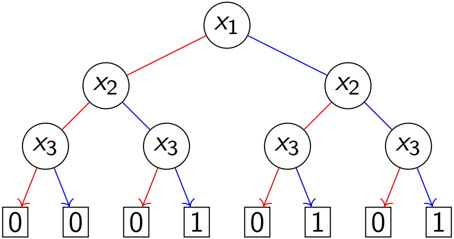
$f|_{a=T} = T \wedge T \wedge (\neg b \vee c) = \neg b \vee c \qquad f|_{a=F} = b \wedge \neg c \wedge T = b \wedge c$
$f|_{a=T,b=F} = T$, $f|_{a=T,b=T} = c$, etc.




```
if a then
  if b then if c then true
                  else false
       else true
else
  if b then if c then true
                  else false
        else false
```

With a fixed variable ordering, the tree is unique: canonical
still *inefficient*: up to 2^n possible combinations, like the truth table

From decision tree to decision diagram



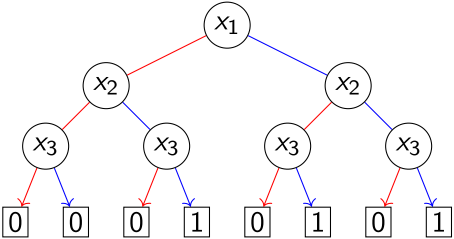
$f(x_1, x_2, x_3) = (\neg x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$
e.g., $f(T, F, T) = T$, $f(F, T, F) = F$, etc.

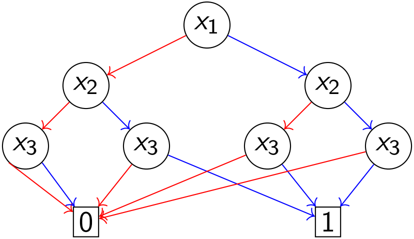
leaf/terminal nodes: function value (0 or 1, i.e, F or T)
inner/nonterminal nodes: x_i (variables the function depends on)
branches: *low*(n) / *high*(n) : assignment F/T of node variable

We define 3 transformation rules for a more compact form:
binary decision diagram.

Reduction nr. 1: Merge leaf nodes

We keep a single copy for nodes 0 and 1

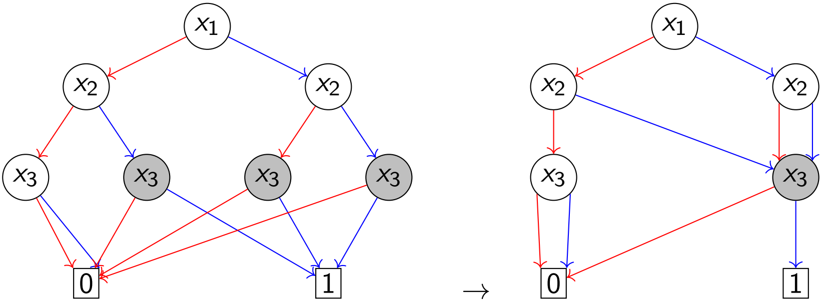




Reduction nr. 2: Merge nodes with same structure

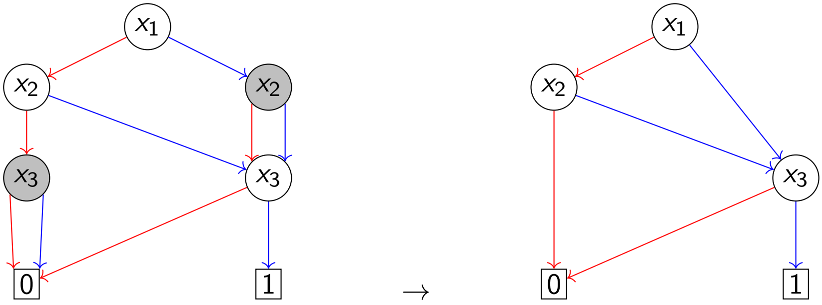
If $\textit{low}(n_1) = \textit{low}(n_2)$ and $\textit{high}(n_1) = \textit{high}(n_2)$, we merge n_1 and n_2

two nodes with equal results on the false branch and equal results on the true branch yield the same value

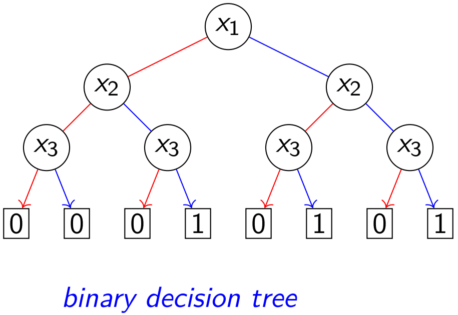


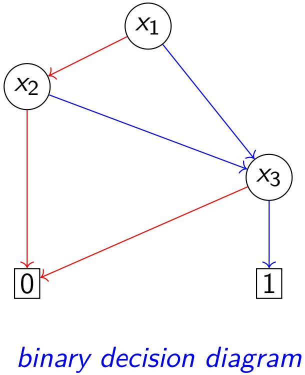
Reduction nr. 3: Eliminate useless tests

Eliminate nodes with the same result on false and true branches



Review: from tree to binary decision diagram





The three reductions are used to *define* a BDD.

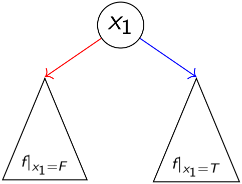
In practice, we want to *avoid* the decision tree due to its size.

We *directly* apply expansion with respect to a variable

Practical BDD construction

~~Don't start from a complete binary tree~~

Build BDD directly, *recursively*, *decomposing* with respect to a variable:



$f = x_1 \wedge f|_{x_1=T} \vee \neg x_1 \wedge f|_{x_1=F}$

build $f|_{x_1=T}$ and $f|_{x_1=F}$

merge any common nodes

BDD libraries: use recursive processing with *lookup/hashing*

- if BDD already exists, it is found and used, not duplicated
- equivalence check is effectively pointer comparison!

Example BDD construction

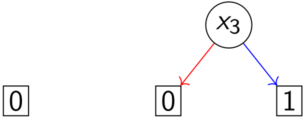
$f(x_1, x_2, x_3) = (\neg x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$

Choose a variable: x_1. Compute $f|_{x_1=F}$ and $f|_{x_1=T}$

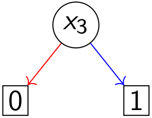
Build BDD for the two functions: directly, if simple (T, F, p, $\neg p$), else *recursively*, choosing *a new variable*:

$f_1 = f|_{x_1=F} = x_2 \wedge x_3$

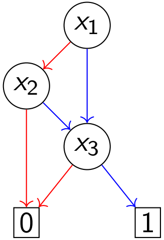
$f_1|_{x_2=F} = \text{F}$ $\qquad f_1|_{x_2=T} = x_3$



$f|_{x_1=T} = x_3$

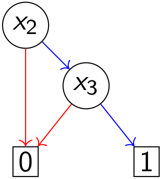


$\downarrow$



Add decision node for x_2

$\downarrow$



Add decision based on x_1

$\longrightarrow$

The BDD rooted at x_3 is common, we keep one copy.

Use of BDDs

Represent Boolean functions *efficiently*
small in many practical cases (still worst-case exponential)

Check equivalence of two Boolean functions

Both are needed in CAD software for *circuit design* (logic synthesis)
apply *optimizations*
then check that result is *correct* (equivalent to original)
(two equivalent circuits/functions must have same BDD)

Frequent use in *formal verification* (check system correctness), efficiently represent large state spaces, etc.

Construct BDDs interactively:
`http://formal.cs.utah.edu:8080/pbl/BDD.php`

Review: Representation Questions

Can we represent a propositional formula in a "systematic" way?

Can we represent propositional formulas uniquely ?

Ideally, a representation should be

simple and *compact* (easy to implement and store)

easy to process (simple, efficient algorithms)

canonical (a formula is represented in a single way)
two formulas are equal precisely if they have same representation