COMPSCI 501: Formal Language Theory

Lecture 4: Regular Expressions

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Regular Expressions

We've seen regular languages are closed under

- Union
- Concatenation
- Kleene Star

Starting from **automata** for the languages, we've constructed *automata* that correspond to these operations.

We can also use notation that directly describes and performs operations on **languages**.

Definition of Regular Expressions

R is a regular expression if it is

- a, for some symbol $a \in \Sigma$ represents $\{a\}$
- ε represents $\{\varepsilon\}$
- $\emptyset$
- $(R_1 \cup R_2)$ where R_1 and R_2 are regular expressions
  sometimes denoted $R_1 + R_2$ or $R_1|R_2$
- $(R_1 \circ R_2)$ where R_1 and R_2 are regular expressions
  sometimes written simply R_1R_2
- (R_1^*) where R_1 is a regular expression

inductive definition (by *structural induction*)

Some definitions have only two base cases; $\varepsilon = \emptyset^*$ is derived

More shorthands: $R^+ = RR^*$

Examples of Regular Expressions

Consider alphabet $\Sigma = \{0, 1\}$. Write Σ for $0 \cup 1$.

Single 1: 0^*10^*
At least a 1: $\Sigma^*1\Sigma^*$
Length $\equiv 1 \mod 3$: $\Sigma(\Sigma\Sigma\Sigma)^*$
Every 0 followed by a 1: $1^*(01^+)^*$ or $(1|01)^*$
Contains no 00: like above, but can end in 0: $(1|01)^*(0+\varepsilon)$
$(0^*1)^*$: does not end in 0

Real numbers with optional sign: $(+|-|\varepsilon)(D^*.D^+|D^+.D^*)$
(must include decimal point, otherwise int)

Used in lexical analysis (compiler)
Recognize email addresses, URLs, etc.

Regular Expression Identities

There are multiple regular expressions describing a given language.

Basic Identities:
$R \cup \emptyset = R$
$R\varepsilon = \varepsilon R = R$

Others:
$(R^*)^* = R^*$
$\varepsilon \cup RR^* = R^*$
etc.

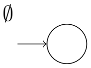
Equivalence with Finite Automata

A language is regular if and only if a regular expression describes it.
(*Kleene's Theorem*)

Proof: by construction
(1) Construct automaton (NFA) from regular expression
(2) Construct regular expression for automaton

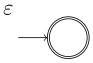
For (1), we start with automata for the three base cases

$\emptyset$



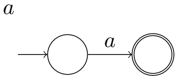
no accepting state

ε



initial state is accepting

a



accepts a

We can apply the construction discussed for NFAs.

Closure under Union

Add new initial state with ε-transitions to both initial states

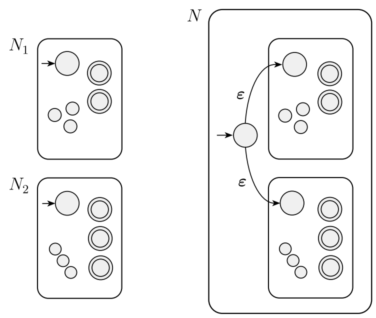


FIGURE 1.46
Construction of an NFA N to recognize $A_1 \cup A_2$

Closure under Concatenation

Add ε-transitions from all accept states of N_1 to initial state of N_2

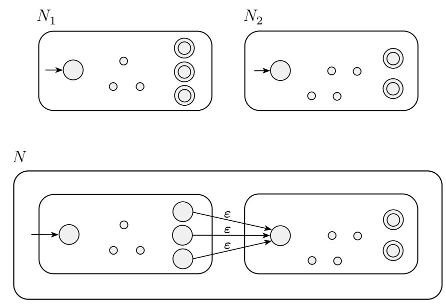


FIGURE 1.48
Construction of N to recognize $A_1 \circ A_2$

Closure under Kleene Star

Add ε-transitions from all accept states to initial state, and
new initial (and accepting) state with ε-transitions to original one.

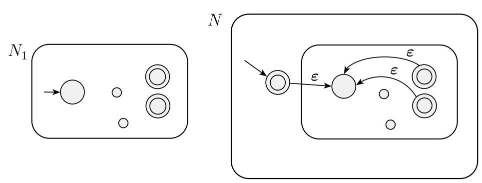


FIGURE 1.50
Construction of N to recognize A^*

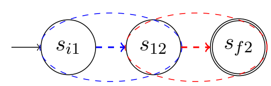
Alternative Construction

We can also maintain these invariants in the construction:

- one initial and (at most) one accepting state
- initial state has no incoming transitions
- accepting state has no outgoing transitions

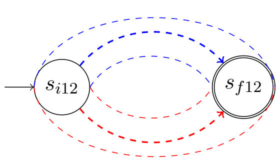
attempting to reduce the number of ε-transitions

Concatenation R_1R_2: merge R_1's accept state and R_2's initial state

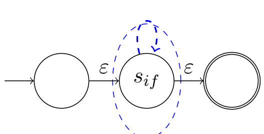


Alternative Constructions (cont'd.)

Union $R_1 \cup R_2$: merge initial and final states



Kleene Star: merge initial and accept state
create new initial and accept state with ε-transitions to loop



Correctness proof: invariants are maintained

Converting Automata to Regular Expressions

Generalized Nondeterministic Finite Automaton (GNFA)

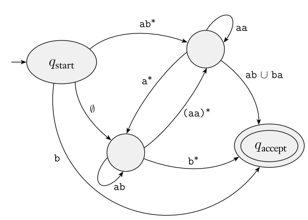


FIGURE 1.61
A generalized nondeterministic finite automaton

- transitions labeled with *regular expressions*
- single initial state, no incoming transitions
- single accept state (not initial), no outgoing transitions

textbook: transitions between all other states (labeled $\emptyset$ if needed)

Eliminating States

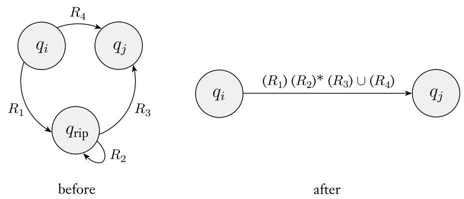


FIGURE 1.63
Constructing an equivalent GNFA with one fewer state

Successively eliminate all states except initial and final

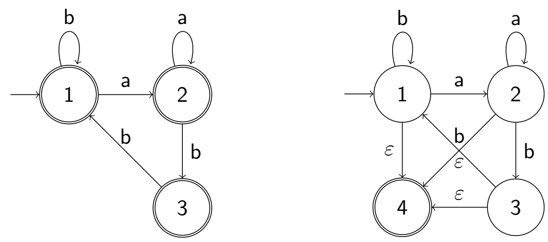
When eliminating state q_{rip}, consider all paths $q_i \to q_{rip} \to q_j$

Augment transition $q_i \stackrel{R_4}{\to} q_j$ with regular expression $R_1 R_2^* R_3$

Finally, only initial and accept states with overall regex left.

Correctness proof: by induction over number of states.

Example: Strings with no *aba*



Introduce new accepting state 4

Eliminate 2, then 3

$R = (b|a^+bb)^*(a^*|a^+b)$

Determining membership

Given a regular expression R and a string u, find whether $u \in R$

Construct automaton for R ("compile" regex), run on u

What is the complexity? worst-case exponential in size of R
DFA may be exponential in size of NFA
or, if we backtrack in NFA, can spend exponential time

ReDoS (denial of service attack): if R controlled by user

- nested repetitions ($^+$ applied to complex expression)
- a match is also a suffix of another match, e.g. $aa...ax \in (a^+)^+$?

⇒ check libraries for evil regexes
do lazy construction of DFA if regex user-supplied

Derivative of Regular Expression

(Brzozowski, 1964)

Q: Given a language L and a symbol a, what are the suffixes of strings in L that start with a ?

$$\partial_a L = \{v \mid av \in L\}$$

Example: $\partial_a a^*b = a^*b$, $\partial_b a^*b = \varepsilon$

How could we use the derivative?

- check string membership: take derivative for successive symbols $a_1, a_2, \ldots a_n$, check if result accepts ε
- directly construct DFA from regular expression!

Derivative Rules

$\partial_a \emptyset = \emptyset$

$\partial_a \varepsilon = \emptyset$

$\partial_a a = \varepsilon$
$\partial_a b = \emptyset$

$\partial_a (R + S) = \partial_a R + \partial_a S$

$\partial_a (RS) = (\partial_a R)S$ if $\varepsilon \notin R$
$\partial_a (RS) = (\partial_a R)S + \partial_a S$ if $\varepsilon \in R$

$\partial_a R^* = \partial_a R \cdot R^*$

Constructing DFA with Derivatives

Every distinct regular expression obtained is a *state*.

A state is accepting if regex contains ε.

Construct DFA for $(a^*b)^*$

$\partial_a (a^*b)^* = \partial_a (a^*b) \cdot (a^*b)^* = a^*b(a^*b)^*$
$\partial_b (a^*b)^* = \partial_b (a^*b) \cdot (a^*b)^* = \varepsilon(a^*b)^* = (a^*b)^*$

$\partial_a (a^*b(a^*b)^*) = \partial_a (a^*b)(a^*b)^* = a^*b(a^*b)^*$
$\partial_b (a^*b(a^*b)^*) = \partial_b (a^*b)(a^*b)^* = (a^*b)^*$