Review: Circuit families and complexity

Circuits are fixed-size \( \Rightarrow \) family of circuits, one per input length

**Def. A circuit family** \( C \) is an infinite list of circuits, 
\( \langle C_0, C_1, C_2, \ldots \rangle \), where \( C_n \) has \( n \) input variables.

\( C \) decides a language \( A \) over \( \{0,1\} \) if for every string \( w, \)
\( w \in A \) iff \( C_n(w) = 1. \) \( (n = |w|) \)

Can define:
- \( \text{size, size-minimal circuits (no smaller equivalent)} \)
- \( \text{size complexity of a family: } f : \mathbb{N} \rightarrow \mathbb{N}, f(n) = \text{size}(C_n) \)
- \( \text{depth, depth minimal, depth complexity (measure of parallel time complexity)} \)

Circuit complexity of a language = size complexity of minimal circuit family

Parallelism: Processor vs. circuit models

*Parallel Random Access Machine* (PRAM):
many (simple) processors simultaneously access shared memory 
with various forms of conflict resolution

Again, PRAM model handles *all input lengths*
For circuits: need to easily obtain entire family

**Def. A family of circuits** \( \langle C_0, C_1, \ldots, C_n \rangle \) is uniform 
if some log-space transducer outputs \( C_n \) when input is \( 1^n \).

**Simultaneous size-depth** circuit complexity \((f(n), g(n))\) if:
- size complexity \( f(n) \)
- depth complexity \( g(n) \)

Why simultaneous?

Examples: poly size, poly-log depth

Insight: can often group circuits in *binary tree*

**Parity:** \([x_1 \oplus \ldots \oplus x_m] \oplus [x_{m/2+1} \oplus \ldots \oplus x_n]\)
size-depth: \((O(n), O(\log n))\)

**Boolean matrix multiplication:** \( c_{ij} = V_{j} \langle a_{ij} \wedge b_{jk} \rangle \)
binary tree for the OR
\( m^3 \) AND gates, \( m^3 \) OR gates, input size \( n = O(m^2) \)
size-depth \((O(n^{3/2}), O(\log n))\)

**Transitive closure:** \( A \vee A^2 \vee \ldots \vee A^m \) for matrix
path of up to \( m \) edges \( \Rightarrow \) pairwise connectivity circuit for \( A \vee A^2 \): size-depth \((O(n^{3/2}), O(\log n))\)
repeated squaring \( A \vee \ldots \vee A^4 \), etc.
\( O(\log n) \) levels of squaring (multiplication with same inputs)
size-depth \((O(n^{3/2} \log n), O(\log^2(n)))\)

The Class NC

**Def.** \( \text{NC}^i \) \( (i \geq 1) \) is the class of languages that can be decided by a 
uniform family of circuits with polynomial size and \( O(\log^i n) \) depth.

\textbf{NC} is the class of languages that are in \text{NC}^i for some \( i \).

Functions implemented by such circuits are \text{NC}^i computable.

Other circuit models (brief mention):
\( \text{AC}^i \): like \text{NC}, but gates have unlimited fan-in (inputs).
\( \text{TC}^i \): like \text{AC}, and also *majority* gates ("threshold circuits").

Can prove (not here): \( \text{NC}^i \subseteq \text{AC}^i \subseteq \text{TC}^i \subseteq \text{NC}^{i+1} \)

Where is NC in hierarchy?

**Theorem:** \( \text{NC} \subseteq \text{P} \)
Do in poly-time:
1. generate circuit \( C_n \)
2. evaluate on \( w \) \((|w| = n)\)

**Theorem:** \( \text{NC}^1 \subseteq \text{L} \)
Can we evaluate a \text{NC}^1 circuit in log space?
1. Construct circuit: doable with log-space transducer (by def.)
2. Evaluate circuit: recursively, from output (DFS)
   need stack (path): ok, since circuit only has depth \( O(\log n) \)

Next: bound in the other direction: \( \text{NL} \subseteq \text{NC}^2 \)
NL ⊆ NC²

Nondeterminism ⇒ must evaluate all paths
⇒ construct (closure of) transition relation

From NL machine $M$, construct configuration graph $G$ of $M$:
   edge for any $c_1 \rightarrow c_2$.
A configuration has log space ⇒ polynomially many
Condition edge $(c_1, c_2)$ on legal values of cell read by head
   0 or 1 (or both)
Can compute path relation (transitive closure of $G$’s edge relation)
by poly-size circuit of $O(\log^2 n)$ depth
Construction doable in log space (see proof that $PATH$ is NL-hard)

P-completeness

Def. A language $B$ is P-complete if
1. $B \in P$ and
2. every language in $P$ is log space reducible to $B$
If $A \leq_L B$ and $B$ is in NC, then $A$ is in NC.

$CIRCUIT$-$VALUE$ problem = evaluating a circuit on its input
$\{\langle C, x \rangle \mid C$ is a circuit and $C(x) = 1\}$.

$CIRCUIT$-$VALUE$ is P-complete
For a TM taking $t(n)$ steps, we’ve constructed an $O(t^2(n))$ circuit
Construction is repetitive (encode next symbol for each cell)
and can be done in log space.