Review: Decidable problems

For regular and context-free languages:

Acceptance problem
Given a pair of finite automaton / regular expression / grammar, and a string, is the string accepted / generated?

Emptiness problem
Is a language defined by a finite automaton / regular expression / grammar empty?

Equivalence Problem
Do two finite automata / regular expressions recognize the same language? decidable

Do two CFGs generate the same language? undecidable

Acceptance Problem for Turing Machines

\[ \mathcal{A}_{TM} = \{ \langle M, w \rangle | M \text{ is a TM that accepts string } w \} \]

\[ \mathcal{A}_{TM} \text{ is Turing-recognizable: simulate } M \text{ on input } w, \text{ accept and reject accordingly.} \]

Universal Turing machine: can simulate any Turing machine
think: general purpose processor

Countable sets

We say two sets \( A \) and \( B \) are the same size if there exists a bijective function \( f: A \to B \) (a correspondence) between them

- injective (one-to-one): \( x \neq y \implies f(x) \neq f(y) \)
- surjective (onto): \( \forall y \in B \exists x \in A : f(x) = y \)

Can now compare sizes also for infinite sets.

A set is countable if it is finite or has the same size as \( \mathbb{N} \).
equivalently: the same size as some subset of \( \mathbb{N} \)
or: the is a one-to-one function from the set to \( \mathbb{N} \)

Diagonalization: Rationals are Countable

\[
\begin{array}{cccccc}
1/1 & 1/2 & 1/3 & 1/4 & \ldots \\
2/1 & 2/2 & 2/3 & 2/4 & \ldots \\
3/1 & 3/2 & 3/3 & 3/4 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

Count elements by diagonal \( (m/n \text{ with } m + n \text{ constant}) \):

- \( 1/1, 1/2, 2/1, 1/3, 2/2, 3/3, 1/4, 2/3, 3/2, 4/1, \ldots \)
- ignore repetitions, they don’t matter

Closure of Countable Sets

The union of two countable sets is countable

A countable union of countable sets is countable

\( C \) countable, \( A_i \) countable for \( i \in C \): \( \bigcup_{i \in C} A_i \) is countable

The cartesian product \( A \times B \) of two countable sets is countable

same as for rationals

If \( A \) is countable, \( A^* = \bigcup_{k \geq 0} A^k \) is countable
### Diagonalization: Reals are Uncountable

Assume reals were countable and enumerate all reals in interval [0, 1) (in decimal notation)

\[
\begin{align*}
  r_1 &= 0. d_{11} d_{12} d_{13} \ldots = 0.12749\ldots \\
  r_2 &= 0. d_{21} d_{22} d_{23} \ldots = 0.2340567\ldots \\
  r_3 &= 0. d_{31} d_{32} d_{33} \ldots = 0.35790856\ldots \\
  \vdots & \quad \vdots \\
\end{align*}
\]

Construct another real number \( \bar{x} = 0.d_1 d_2 d_3 \ldots \) digit by digit: let \( d_i \) be different from \( d_{ii} \) (\( i \text{th} \) digit of \( i \text{th} \) number).

For instance, take \( d_i = d_{ii} - 1 \) if \( d_{ii} > 0 \) and \( d_i = 1 \) if \( d_{ii} = 0 \) (avoid duplicate representations with 9: \( 0.19999\ldots = 0.2 \))

We get a number \( \bar{x} \) which is not in the sequence

\[ \Rightarrow \text{ reals are uncountable} \]

### Cantor’s Theorem: \( |X| < |P(X)| \)

There is no bijection between any set and its powerset.

Easy to see for finite sets, \( n < 2^n \). In general:

Assume a bijection \( f: X \rightarrow P(X) \) exists.

Construct the set \( Y = \{ x \in X \mid x \notin f(x) \} \)

Since \( Y \in P(X) \), and \( f \) is bijective, there is \( y \in X \) with \( f(y) = Y \).

We examine whether \( y \in Y \), or equivalently, \( y \notin f(y) \).

If \( y \in Y \), since \( Y = f(y) \), we get \( y \notin f(y) \), so \( y \) does not observe the required condition for \( Y \), contradiction.

If \( y \notin Y \), then \( y \notin f(y) \) so \( y \) observes the condition for \( Y \), and we should have \( y \in Y \), contradiction.

Thus a bijection cannot exist.

### The set of all languages is uncountable

From Cantor’s theorem, \( \Sigma^* \) countable, \( P(\Sigma^*) \) is not.

Or: The set \( B \) of all infinite binary sequences is uncountable \( \{0,1\}^\omega \) : corresponds to set of all real numbers in \([0, 1]\)

Let \( \mathcal{L} = P(\Sigma^*) \) be the set of all languages over \( \Sigma \).

Construct a bijection from \( \mathcal{L} \) to \( B \).

Let \( \Sigma^* = \{ s_1, s_2, s_3, \ldots \} \).

Construct characteristic sequence of language \( A \in \Sigma^* \):

\[ \text{bit } b_i = 1 \text{ if } s_i \in A \text{, and } b_i = 0 \text{ if } s_i \notin A. \]

Clearly this is a bijection. So \( \mathcal{L} \) is uncountable.

There are more languages than Turing machines

\( \Rightarrow \) some languages are not recognizable (thus also undecidable).

### Connection to Diagonalization

\[
\begin{array}{cccccccc}
  & (M_1) & (M_2) & (M_3) & (M_4) & \cdots & (D) & \cdots \\
M_1 & \text{accept} & \text{reject} & \text{accept} & \text{reject} & & \text{accept} & \cdots \\
M_2 & \text{accept} & \text{accept} & \text{accept} & \text{accept} & & \text{accept} & \cdots \\
M_3 & \text{reject} & \text{reject} & \text{reject} & \text{reject} & & \text{accept} & \cdots \\
M_4 & \text{accept} & \text{accept} & \text{accept} & \text{accept} & & \text{accept} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \text{reject} & \cdots \\
D & \text{reject} & \text{reject} & \text{accept} & \text{accept} & & \text{accept} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \text{accept} & \cdots \\
\end{array}
\]

**FIGURE 4.21**

If \( D \) is in the figure, a contradiction occurs at “?”

### A Turing-Unrecognizable Language

Recognizing = accept for sure; may or may not reject

Reverse roles: A language is co-Turing-recognizable if it is the complement of a Turing-recognizable language.

**Theorem:** A language is decidable if it is Turing-recognizable and co-Turing-recognizable.

“⇒”: \( A \) decidable \( \Rightarrow \) \( \overline{A} \) decidable (reverse accept/reject)
\( \Rightarrow \) both are recognizable

“⇐”: Let \( M_1 \) be recognizer for \( A \) and \( M_2 \) recognizer for \( \overline{A} \).

run \( M_1 \) and \( M_2 \) in parallel
if \( M_1 \) accepts, accept; if \( M_2 \) accepts, reject (one will happen)

**Corollary:** \( \overline{A}_{TM} \) is not Turing-recognizable.
If it were, since \( A_{TM} \) is recognizable, it would be decidable!