COMPSCI 501: Formal Language Theory

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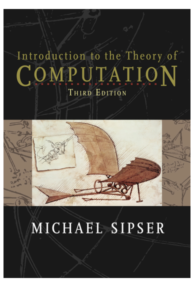
COMPSCI 501: Formal Language Theory

- **Instructor**: Marius Minea, office hous: Tue 3-4 pm, LGRC A261
- **Lectures**: Goessmann 20, MWF 11:15 - 12:05
- **TA**: Rik Sengupta, office hours: Wed 6-7:30 pm, LGRT 220
- **Graders**: Deeksha Razdan, Gourav Saha

Textbook



It's a **great** book

Intuition and in-depth understanding of covered topics

Aim to read chapters *ahead* of lecture

Why study formal language theory?

For some: for the beauty of theory

Understand tools used in practice
regular expressions/automata for string search
grammars: design programming or domain-specific languages

Understand complexity and limits of computation

Improve ability to analyze and solve problems

Motivation: Halting problem

There is no algorithm that can decide whether an arbitrary program halts on a given input.

More formally: Determining whether a *Turing machine* halts on a given input is *undecidable*.

Crucial to establish the limits of computation.

If we are given a (computer science) problem, is it even solvable?

Example: Computer Viruses

A virus can be formally defined as a program that replicates itself.

Is it possible to build the perfect antivirus?
(that detects any virus, with no false positives/negatives)

Fred Cohen showed in 1987 that the problem was undecidable.

The program P under scrutiny could invoke any proposed decision procedure D and infect other programs only if D determines that P is *not* a virus.

Fred Cohen: Computer Viruses: Theory and Experiments, 1987

Example: Optimizing compilers

Is it possible to build the perfect optimizing compiler?
(compiles a program to the shortest/simplest code)

If so, the optimized program should just
(1) read (part of) the input
(2) halt, or loop forever
thus it would have solved the halting problem!

Many fundamentally interesting problems in program analysis/testing/verification are undecidable for the same reason.

Big Picture: Automata, Computability and Complexity

- Complexity Theory

What makes some problems computationally hard and others easy?

Classification scheme according to computational difficulty

Coping with complexity / change problem for easier solution

- Computability Theory

Limits of what can be computed

Limits of what can be proved

- Automata Theory

Formally define simple models of computation finite automata
context-free grammars / pushdown automata

Selective outline

- Deterministic / nondeterministic automata, regular expressions
- Context-free grammars / pushdown automata
- Turing machines
- (Un)decidability, reducibility
- Complexity
  - time
  - space: polynomial, logarithmic
  - impact of nondeterminism
- More advanced topics
  - alternation, games, circuit complexity

Logistics and Grading

- **Homework**: 30% (six homeworks)
- **Midterm 1**: 20% (Thu Feb 21, 7 pm, ILC S131)
- **Midterm 2**: 20% (Wed Apr 10, 7 pm, ILC S131)
- **Final**: 25% (Thu May 9, 10:30 am, Goessmann 20)
- **Moodle quizzes**: 5% (throughout semester)

Review: Sets

- Sets: unordered, unique elements
  - multisets, if number of occurrences matters $\{2, 2, 3, 7, 7, 7\}$
  - *finite* vs. *infinite* sets (results may differ!)
  - *power set* $\mathcal{P}(A)$ = set of subsets of A ($2^{|A|}$ subsets)
  - Cartesian product / cross product:
    $A \times B = \{(x, y) \mid x \in A, y \in B\}$

Functions (mappings)

$f : A \to B$ A = domain, B = range

injective: $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$

surjective (onto): $\forall y \in B \; \exists x \in A \; . \; f(x) = y$

bijective (one-to-one correspondence): injective *and* surjective

Relations

Set of tuples $\subseteq A_1 \times A_2 \times \ldots \times A_k$
(k-ary relation, k-tuples)

A relation defines a *predicate* over $A_1 \times A_2 \times \ldots \times A_k$
true if tuple in relation, false if not

Binary relation: set of pairs $\subseteq A \times B$

Binary relation *on* a set A: subset of $A \times A$

Equivalence relation on a set:
reflexive: $\forall x : R(x, x)$
symmetric: $\forall x, y : R(x, y) \rightarrow R(y, x)$
transitive: $\forall x, y, z : R(x, y) \wedge R(y, z) \rightarrow R(x, z)$

An equivalence relation *partitions* a set into disjoint subsets
e.g., remainders $\mod n$, strongly connected components

Strings and Languages

Alphabet (usually Σ): any nonempty finite set

String: *finite* sequence of symbols from the alphabet

Σ^*: set of all (finite) strings over Σ, incl. *empty string* (denoted ϵ).

Important: distinguish between:
finite: all strings in Σ^* are finite
bounded/unbounded: strings can be arbitrarily long
infinite: infinite strings are *not* in Σ^*

lexicographic order (dictionary order)

shortlex order: ϵ, 0, 1, 00, 01, 10, 11, 000, ...
useful for *enumerating* strings

A **language** is an arbitrary subset of strings (of Σ^*)

Preview: Decidability

A **program** can be viewed (encoded) as a **string**.

A (decision) problem can be viewed as a **set of strings** (inputs) for which the answer is YES.

Cantor's theorem says there is *no* one-to-one mapping (bijection) between a set and its powerset.
the powerset has "more" elements

Thus, there can be no one-to-one mapping between Σ^* (a superset of all programs) and $\mathcal{P}(\Sigma^*)$ (the set of all problems).
$\Rightarrow$ some problems must be undecidable

Graphs

$G = (V, E)$: nodes/vertices, edges

directed / undirected: edges are ordered / unordered pairs

degree/in-/outdegree: number of edges (total/entering/leaving) node

path: sequence of nodes connected by edges (incl. empty)

strongly connected: a directed path between any two nodes

Eulerian path/cycle: contains each *edge* exactly once

Hamiltonian path/cycle: contains each *vertex* exactly once

Logic and proofs

Proof
informally: a convincing logical argument that a statement is true
formally: a sequence of true statements, which are either axioms, hypotheses, or are obtained from previous statements using *deduction rules*

Book gives clear convincing arguments without excessive formalism

But we must still practice being careful that all our steps are correct.

How to prove it ?

- Be patient
- Come back to it
  let it sink in, let ideas develop
- Be neat
  define notions clearly, be explicit about any deductions
- Be concise
  the most beautiful proofs are often short and simple

Proof by construction

Def.: A graph is k-regular if every node has degree k.

Prove: For any even $n > 2$ there is a 3-regular graph with n nodes.

Idea: try a "regular" construction: from each node, an edge going "left", "middle", and "right".

Label nodes $V = \{0, 1, \ldots, n-1\}$. Visualize on a circle.

Construct edges:
$(i-1, i)$ (for $i > 0$) and $(n-1, 0)$, (left/right)
and edges $(i, i + n/2)$ for $i < n/2$ (middle, to opposite nodes).

Proof by construction (2)

Prove: There exist irrational numbers x and y so that x^y is *rational*.

Finding a pair of numbers is non-obvious.

The first irrational number we usually learned of is $\sqrt{2}$.
Why is it irrational ?

Is $\sqrt{2}^{\sqrt{2}}$ rational? Don't know.

If it were (case 1), we have our x and y.

Now assume $\sqrt{2}^{\sqrt{2}}$ irrational. We have
$\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$ (rational), so again we have our two numbers: $\sqrt{2}^{\sqrt{2}}$ and $\sqrt{2}$.

We've shown x and y must exist, without concretely finding them.

Proof by contradiction

To prove P, assume $\neg P$ and derive a contradiction.

Closely related: *proof by contrapositive* (*indirect proof*)

Contrapositive of $P \to Q$ is $\neg Q \to \neg P$.
The two are equivalent.

We could show $\neg Q \to \neg P$ (by direct proof) and can then claim $P \to Q$.

We also have $\neg(P \to Q) = P \wedge \neg Q$.

Thus, if we assume P and $\neg Q$ and derive a contradiction, we have proved $P \to Q$.

Proof by induction

Prove that all elements of an *infinite* set have a given property
typically a predicate $P(n)$ over naturals

Ordinary induction

If $P(n_0)$ holds (typically $n_0 = 0$, or 1, etc.)
and $\forall n \geq n_0 : P(n) \to P(n+1)$
then $\forall n \geq n_0 : P(n)$.

Strong induction
allows us to assume P(n) for *all* smaller values, not just previous:

If $P(n_0)$ holds
and $(\forall k : n_0 \leq k \leq n \to P(k)) \to P(n+1)$
then $\forall n \geq n_0 : P(n)$.

Where is the fallacy?

Prove: In any set of n horses, all have the same color.

Base case: $n = 1$. One horse, clearly the same color.

Inductive step: Assume true for $n \geq 1$, prove for $n+1$.

Remove horse x from set of $n+1$ horses. Remaining set has n horses, all of same color.

Now add back x and remove a different horse y. Again n horses, all of same color.
Thus, adding y back we get $n+1$ horses of same color, q.e.d.

Proof exercise: Ramsey's Theorem

Def. A *clique* in a graph G is a subgraph in which any two nodes are connnected by an edge.

An *anti-clique* (*independent set*) is a subgraph in which any two nodes are *not* connected by an edge.

Prove: Any graph with n nodes contains either a clique or an anti-clique with at least $\frac{1}{2}\log_2 n$ nodes.

Intuition: $\log_2 n$ suggests we need to successively halve the number of nodes.

For next time

- Review notions in intro chapter

(be familiar with terms in one-page glossary)

- Revisit any less familiar notions from CS 250 and 311
- Sign up for Piazza and Gradescope
- Review knowledge on finite automata