Algorithm Design—Greedy

Greedy: make a single “greedy” choice at a time, don’t look back.

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Focus is on proof techniques

- Last time: “greedy stays ahead” (inductive proof)
- This time: exchange argument

Scheduling to Minimize Lateness

- You have a very busy month: $n$ assignments are due, with different deadlines

Assignments:

- $1: |---|$ (len=1, due=2)
- $2: |---|---|$ (len=2, due=5)
- $3: |---|---|---|$ (len=3, due=6)
- $4: |---|---| $ (len=2, due=7)

Deadlines:

- $d_1$ d2 d3 d4
- 0 1 2 3 4 5 6 7 8 9

- How should you schedule your time to “minimize lateness”?

Clicker Question 1

True or false: an algorithm to minimize maximum lateness will also find a schedule that is not late, if one exists.

A. False, because the lateness function is not linear
B. False, because it minimizes the maximum lateness, whereas we want all jobs to have lateness zero
C. True

Possible Greedy Approaches

- **Note:** scheduling work back-to-back (no idle time) can’t hurt
  => schedule determined just by order of assignments

| 1: |---| | (len=1, due=2) |
| 2: |---|---| | (len=2, due=5) |
| 3: |---|---|---| | (len=3, due=6) |
| 4: |---|---| | (len=2, due=7) |

- What order should we choose?
  - **Shortest Length:** ascending order of $t_j$.
  - **Smallest Slack:** ascending order of $d_j - t_j$.
  - **Earliest Deadline:** ascending order of $d_j$.

  - **Earliest deadline first** is better in example. Let’s prove it is always optimal.
Clicker Question 2

If two jobs have the same deadline, the earliest deadline first algorithm should schedule
A. The longest job first, because then its lateness will be minimized
B. The shortest job first, because that has a higher chance of finishing before the deadline
C. Does not matter

Identical Maximum Lateness

Claim: If in a schedule we swap two jobs with the same deadline, we get the same maximum lateness. True?

Not necessarily, if the jobs are not in earliest deadline first order! (Example)

Claim: If in an EDF schedule, we swap two jobs with the same deadline, we get the same maximum lateness.

Proof: Since the schedules are EDF, all jobs with the same deadline are scheduled in a consecutive block. Then among those, the last one has the maximum lateness. That finishing time does not change by swapping schedules within the block.

Corollary All EDF schedules have the same maximum lateness.

Exchange Argument (False Start)

Assume jobs ordered by deadline $d_1 \leq d_2 \leq \ldots \leq d_n$, so the greedy ordering is simply $A = 1, 2, \ldots, n$

Claim: $A$ is optimal

Proof attempt: Suppose for contradiction that $A$ is not optimal. Then, there is an optimal solution $O$ with $O \neq A$

$\Rightarrow$ Since $O \neq A$, there must be two jobs $i$ and $j$ that are out of order in $O$ (e.g. $O = 1, 3, 2, 4$)

$\Rightarrow$ Let's swap $i$ and $j$ and show we get a better solution $O'$

$\Rightarrow$ $A$ is not optimal. Contradiction, so $A$ must be optimal.

Problem? $O'$ may still be optimal. Example?
Can't do proof by contradiction in this way.

Exchange Argument (Correct)

Suppose $O$ optimal and $O \neq A$. Then we can modify $O$ to get a new solution $O'$ that is:

1. No worse than $O$
2. Closer to $A$ is some measurable way

$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \ldots \rightarrow A(\text{optimal})$

High-level idea: gradually transform $O$ into $A$ without hurting solution, thus preserving optimality.
Concretely: show 1 and 2 above.

Exchange Argument for Scheduling to Minimize Lateness

Recall $A = 1, 2, \ldots, n$.

For $S \neq A$, say there is an inversion if $i$ comes before $j$ but $j < i$ (thus $d_j \leq d_i$)

Claim: if $S$ has an inversion, $S$ has a consecutive inversion—one where $i$ comes immediately before $j$. Why?

Main result: Let $O \neq A$ be an optimal schedule. Then $O$ has a consecutive inversion $i, j$.

We can swap $i$ and $j$ to get a new schedule $O'$ such that:

1. $O'$ has one less inversion than $O$
2. Maximum lateness of $O'$ is at most maximum lateness of $O$

Proof:

1. Obvious
2. Next slide(s)

Proof (Lateness does not increase)

Swapping a consecutive inversion ($i$ precedes $j$; $d_j \leq d_i$)

Consider the lateness $\ell'_k$ of each job $k$ in $O'$:

$\Rightarrow$ If $k \notin \{i, j\}$, then lateness is unchanged: $\ell'_k = \ell_k$
$\Rightarrow$ Job $j$ finishes earlier in $O'$ than $O$: $\ell'_j \leq \ell_j$
$\Rightarrow$ Finish time of $i$ in $O'$ = finish time of $j$ in $O$. Therefore

$\ell'_i = f'_i - d_i = f_j - d_i \leq f_j - d_j = \ell_j$

Conclusion: $\max_k \ell'_k \leq \max_k \ell_k$. Therefore $O'$ is still optimal.
For any optimal $O \neq A$ we showed that we could transform $O$ to $O'$ such that:

1. $O'$ is still optimal
2. $O'$ has one less inversion than $A$

$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \ldots \rightarrow A(\text{optimal})$

Since there are at most $\binom{n}{2}$ inversions, by repeating the process a finite number of times we see that $A$ is optimal.

Consider the total lateness $\ell'_i + \ell'_j$ in the new schedule. Which fact about total lateness follows from our argument?

A. It is no more than $2\ell_i$
B. It is no more than $2\ell_j$
C. It is no more than $\ell_i + \ell_j$
D. None of the above

C would imply EDF is also optimal for minimizing total lateness. It is not. There is no known polynomial time algorithm for that.

Optimal offline caching: farthest-in-future (clairvoyant algorithm)

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

Theorem. [Bélády 1966] FF is optimal eviction schedule.

Pf. Algorithm and theorem are intuitive; proof is subtle.

Wrap-Up: Greedy Algorithms

Greedy: make a single "greedy" choice at a time, don’t look back.

| Formulate problem | ✓ |
| Design algorithm | ✓ |
| Prove correctness | ✓ |
| Analyze running time | ✓ |

Specific algorithms Dijkstra, MST

Proof techniques

- Last time: “greedy stays ahead” (inductive proof)
- This time: exchange argument

Need to formulate precisely; careful not to get arguments wrong!