Instructions. You may work in groups, but you must individually write your solutions yourself. List your collaborators on your submission.

If you are asked to design an algorithm as part of a homework problem, please provide: (a) the pseudocode for the algorithm, (b) an explanation of the intuition for the algorithm, (c) a proof of correctness, (d) the running time of your algorithm and (e) justification for your running time analysis.

Submissions. Please submit a PDF file. You may submit a scanned handwritten document, but a typed submission is preferred. It will be very helpful if in your submission each question starts on a new page.

1. (20 points) Updating flows Let $G = (V, E)$ be a unit-capacity flow network with source $s$ and sink $t$. We are also given an integer maximum flow for $G$. Describe how the maximum flow can be efficiently updated when a new edge with unit capacity is added to $E$; b) deleted from $E$.

2. (20 points) Reduce Flows (K&T Ch.7 Ex.12) You are given a flow network with unit-capacity edges: It consists of a directed graph $G = (V, E)$, a source $s \in V$, and a sink $t \in V$; and $c_e = 1$ for every $e \in E$. You are also given a parameter $k$. The goal is to delete $k$ edges so as to reduce the maximum $s - t$ flow in $G$ by as much as possible. In other words, you should find a set of edges $F \subset E$ so that $|F| = k$ and the maximum $s - t$ flow in $G' = (V, E \setminus F)$ is as small as possible subject to this. Give a polynomial-time algorithm to solve this problem.

3. (15 points) Edge-disjoint paths Given a directed graph $G = (V, E)$ with vertices $s, t \in V$, give an algorithm that finds the maximum number of edge-disjoint paths from $s$ to $t$.

Extra credit (10 points): Do the same for node-disjoint paths.

4. (25 points) Escape problem (K&T Ch.7 Ex.14)

Consider a directed graph $G = (V, E)$ and two disjoint sets of nodes $X, S \subset V$. A set of evacuation routes is a set of paths so that (i) each node in $X$ is the start of one path, (ii) each path ends at a node in $S$, and (iii) the paths do not share any edges.

a) Show how to decide in polynomial time whether a set of escape routes exists.

b) Do the same when (iii) reads “the paths do not share any nodes”.

c) Give an example with the same $G, X$ and $S$ when (a) is possible but (b) not.

5. (20 points) Minimum Path Cover (CLRS P. 26-2)

A path cover of a directed graph $G = (V, E)$ is a set $P$ of vertex-disjoint paths such that every vertex in $V$ is included in exactly one path in $P$. Paths may start and end anywhere, and they may be of any length, including 0. A minimum path cover of $G$ is a path cover containing the fewest possible paths.

a) Give an efficient algorithm to find a minimum path cover of a directed acyclic graph $G = (V, E)$. Hint: Assuming that $V = \{1, 2, \ldots, n\}$, construct and run a maximum flow algorithm on the graph $G' = (V', E')$, where $V' = \{x_0, x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_n\}$, $E' = \{(x_0, x_i) : i \in V\} \cup \{(x_i, y_i) : (i, j) \in E\}$. b) Does your algorithm work for directed graphs that contain cycles? Prove or give a counterexample.

6. (0 points). How long did it take you to complete this assignment?