COMPSCI 311 Introduction to Algorithms

Lecture 4: Graphs: BFS and DFS

Marius Minea

University of Massachusetts Amherst

slides credit: Dan Sheldon, Akshay Krishnamurthy, Andrew McGregor

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Review: BFS

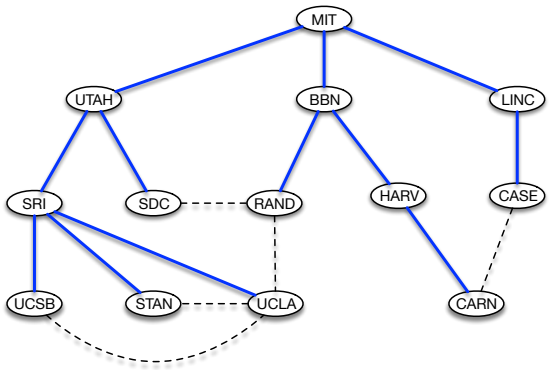
BFS(s):

```
mark s as "discovered"                                   ▷ 1
L[0] ← {s}, i ← 0                                        ▷ 1
while L[i] is not empty do
    L[i+1] ← empty list                                  ▷ ≤ n
    for all nodes v in L[i] do                           ▷ n
        for all neighbors w of v do                      ▷ 2m
            if w is not marked "discovered" then         ▷ 2m
                mark w as "discovered"                   ▷ n
                put w in L[i+1]                          ▷ n
            end if
        end for
    end for
    i ← i+1                                              ▷ ≤ n
end while
```

Running time: $O(m+n)$.
Hidden assumption: can iterate over neighbors of v efficiently.

BFS Tree

We can use BFS to make a tree. (blue: "tree edges", dashed: "non-tree edges")

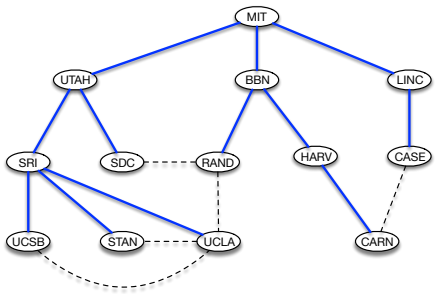


BFS Tree

BFS(s):

```
mark s as "discovered"
L[0] ← {s}, i ← 0
T ← empty
while L[i] is not empty do
    L[i+1] ← empty list
    for all nodes v in L[i] do
        for all neighbors w of v do
            if w is not marked "discovered" then
                mark w as "discovered"
                put w in L[i+1]
                put (v, w) in T
            end if
        end for
    end for
    i ← i+1
end while
```

BFS Tree



Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G.
Then the layers of x and y in T differ by at most 1.

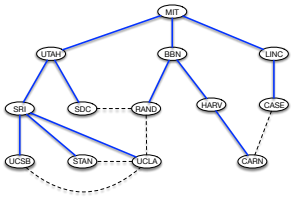
BFS and non-tree edges

Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G.
Then the layers of x and y in T differ by at most 1.

Proof

- Let (x, y) be an edge
- Assume x is discovered first and placed in L_i
- Then $y \in L_j$ for $j \geq i$
- When neighbors of x are explored, y is either already in L_i or L_{i+1}, or is discovered and added to L_{i+1}

Clicker Question 1



Suppose in BFS that the nodes in each layer are explored in a different order (e.g. reverse). Which of the following are true?

A. The nodes that appear in each layer may change

B. The BFS tree may change

C. Both A and B

D. Neither A nor B

BFS and Connectivity

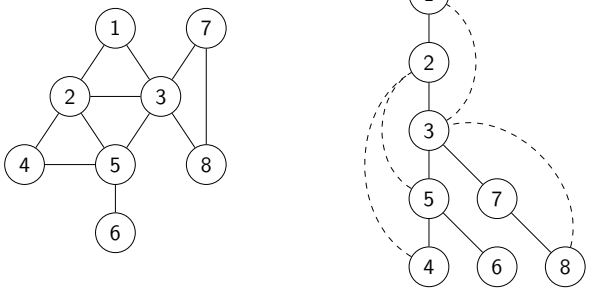
Can we use BFS to detect cycles ?

What if we find an already **discovered neighbor**?
only if it is **not the parent** (same edge backwards)

for all nodes v in $L[i]$ **do**
  for all neighbors w of v **do**
    if w is not marked "discovered" **then**
      mark w as "discovered"
      put w in $L[i]$
      $parent[w] = v$
    else if $w \neq parent[v]$ **then**
      output "graph has cycle"
    end if
  end for
end for

Depth-First Search

Depth-first search (DFS): keep exploring from the most recently added node until you have to backtrack.



Dotted edges: to already explored nodes

DFS: Recursive Implementation

DFS(u)
  mark u as "explored"
  for all edges (u, v) **do**
    if v is not "explored" **then**
      call DFS(v) recursively
    end if
  end for

DFS: Running Time

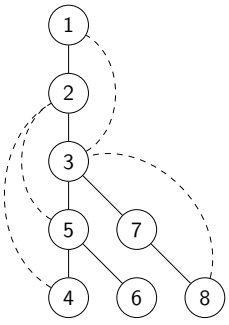
How to analyze if algorithm is recursive?
Same: count executions of each line, *including* recursive call

DFS(u)
  mark u as "explored" ▷ n
  for all edges (u, v) **do** ▷ $2m$
    if v is not "explored" **then** ▷ $2m$
      call DFS(v) recursively ▷ n
    end if
  end for

Running time: $O(m + n)$ same complexity as BFS
Same assumptions: can traverse neighbor list in time proportional to node degree

DFS Tree

$T \leftarrow$ empty
DFS(u)
  mark u as "explored"
  for all edges (u, v) **do**
    if v is not "explored" **then**
      put (u, v) in T
      call DFS(v) recursively
    end if
  end for



Claim: Non-tree edges lead to (indirect) **ancestors**

DFS: Non-tree edges lead to ancestors

Claim: Let T be the tree discovered by DFS, and let (x, y) be an edge of G that is not in T. Then x or y is an ancestor of the other.

Proof:

- Let x be the first of the two nodes explored
- Is y explored at beginning of DFS(x)? No.
- At some point *during* DFS(x), we examine the edge (x, y). Is y explored then? Yes, otherwise we would put (x, y) in T
- $\Rightarrow$ y was explored *during* DFS(x)
- $\Rightarrow$ y is a descendant of x

We first see the edge as (y, x) when exploring y: ancestor x is already marked explored, so (y, x) is a **back edge**.

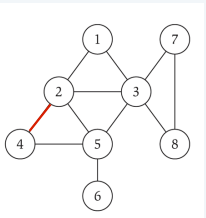
x can't be parent of y, since then (x, y) is a **tree edge**.

Graph Representation

- What data structure do we use to represent a graph?
- How do we iterate through nodes, edges, neighbors?
- Has impact on memory efficiency and running time.

Graph Representation: Adjacency Matrix

n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge (symmetric)

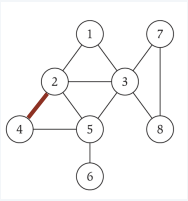


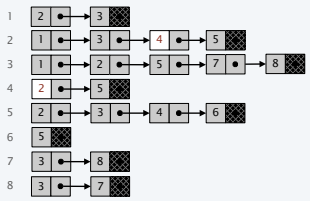
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Space proportional to n^2
- Checking is (u, v) is an edge takes $\Theta(1)$ time (lookup)
- Iterating through all neighbors takes $\Theta(n)$
- Iterating through all edges takes $\Theta(n^2)$

Graph Representation: Adjacency Lists

Each node keeps **list of neighbors**





- Each edge stored twice
- Space? $\Theta(m + n)$
- Checking if (u, v) is an edge? $O(\text{degree}(u))$ time (degree = number of neighbors)

Traversal Implementations

Generic approach: maintain set of explored nodes and discovered nodes

- Explored = have seen this node and explored its outgoing edges
- Discovered = the "frontier". Have seen the node, but not explored its outgoing edges.

Generic Graph Traversal

Let A = data structure of discovered nodes

Traverse(s)
  put s in A
  while A is not empty **do**
    take a node v from A
    if v is not marked "explored" **then**
      mark v "explored"
      for each edge (v, w) incident to v **do**
        put w in A ▷ w is discovered
      end for
    end if
  end while

BFS: A is a queue (FIFO) DFS: A is a stack (LIFO)

Can a node be discovered (placed in A) multiple times? Yes.
For DFS, node is explored from parent that added it last (LIFO).
For BFS, can avoid by not adding discovered nodes.

Clicker Question 2

put s in A
while A is not empty **do**
  take a node v from A
  if v is not marked "explored" **then**
    mark v "explored"
    for each edge (v, w) incident to v **do**
      put w in A ▷ w is discovered
    end for
  end if
end while

What is the maximum number of times a node w can be put in A?

- A: once
- B: $degree(w) + 1$ times
- C: $2 \cdot degree(w)$ times
- D: $|V|$ times

Clicker Question 3

DFS(u)
  Mark u as "explored"
  for each edge (u, v) **do**
    if v is not "explored" **then**
      Call DFS(v) recursively
    end if
  end for

Put s in A
while A is not empty **do**
  Take a node v from A
  if v is not "explored" **then**
    Mark v as "explored"
    for each edge (v, w) **do**
      Put w in A
    end for
  end if
end while

Suppose we have a tree with n nodes, height h and degree d. Compare recursive and non-recursive DFS in terms of memory used for the stack

- A: recursive: $\Theta(hd)$, non-recursive: $\Theta(h)$
- B: recursive: $\Theta(h)$, non-recursive: $\Theta(hd)$
- C: recursive: $\Theta(h)$, non-recursive: $\Theta(d)$
- D: recursive: $\Theta(n)$, non-recursive: $\Theta(hd)$

Exploring *all* Connected Components

How to explore entire graph even if it is disconnected?

while there is some unexplored node s **do**
  Traverse(s) ▷ Run BFS/DFS starting from s.
  Extract connected component containing s
end while

Running time? Does it change?

Naive: $O(m + n)$ per component $\Rightarrow O(c(m + n))$ if c components.

Better: Search on component C only works on nodes/edges in C

- Time for component C: $O(\#\text{edges in } C + \#\text{nodes in } C)$
- $O(n)$ to detect all components
- Total time: $O(m + n)$

Usually OK to assume graph is connected.
State if you are doing so and why it does not trivialize the problem.

Review and Outlook

- Graph traversal by BFS/DFS
  - Different versions of general exploration strategy
  - $O(m + n)$ time
  - Produce trees with useful properties (for other problems)
  - Basic algorithmic primitive — used in many other algorithms path from s to t, connected components
- Bipartite testing
- Directed graphs
  - Traversal
  - Strong connectivity
  - Topological sorting