# COMPSCI 311: Introduction to Algorithms

Lecture 25: Randomized Algorithms

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## Deterministic Algorithms

- So far: deterministic algorithms on worst case inputs.
- Why deterministic algorithms?
  - Easier to understand, pretty powerful.
- Why worst case?
  - Enables precise statements
  - But maybe not reflective of real-world instances.
  - Average-case analysis? What distribution?

## Why Randomized Algorithms ?

- Efficient **in expectation**
- Optimal with **high probability**
- Break (undesired) symmetry
- Show some solution exists, or derive bound on their number

Types of randomized algorithms:

- Fail with some small probability
- Always succeed, but running time may be non-polynomial

Examples

- Min-Cut
- Randomized Median Finding
- Max 3-SAT

## Minimum Cuts

**Problem.** Given undirected $G = (V, E)$, partition $V$ into sets $A, V \setminus A$, minimizing edge count across cut:

$$\text{cut}(A) = |\{(u, v) \in E, u \in A, v \notin A\}|$$

- We saw how to compute minimum $s - t$ cut in directed graph.
  for *fixed* $s$, $t$
- How do we compute *global minimum cut*? Is this harder?

## Deterministic Algorithm

**Idea.** Convert into $s - t$ cut in directed graph.

Replace $e = (u, v)$ with directed edges both ways (capacity 1).
Pick arbitrary $s$.
**for** each other vertex $t$ **do**
  Compute minimum $s - t$ cut.
Return smallest computed $s - t$ cut.

**Running Time.** $n$ max-flow computations $\Rightarrow O(mn^2)$ at best.

## Contraction Algorithm (Karger, 1995)

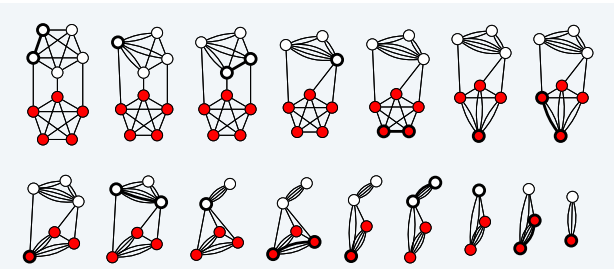
- Idea: only edges across cut matter, edges within set don't
- Collapse $S$ and $V \setminus S$ into one node each
- But allow multiple edges between nodes (*multigraph*)

**Def.** Multigraph $G = (V, E)$ is a graph that can have parallel edges.
**Def.** Contracting an edge $(u, v)$ in $G = (V, E)$ produces a new multigraph $G' = (V', E')$

- With new node $w$ instead of $u, v$ (edges $(u, v)$ deleted).
- If $(x, u)$ or $(x, v) \in E$, then $(x, w) \in E'$.
- All other edges preserved.

## Contraction Algorithm: Idea

- Pick edge $(u, v) \in E$ uniformly at random
- Contract edge $(u, v)$, replacing $u$, $v$ with new node $w$
- Preserve edges, setting $u$ and $v$ endpoints to $w$
- Keep multiple edges, delete self-loops



Reference: Thore Husfeldt

image credit: Kevin Wayne / Pearson

## Contraction Algorithm

$S(v) = \{v\}$ for all $v \in V$. ▷ nodes contracted into $v$
**while** $|V| > 2$ **do**
  Pick edge $(u, v) \in E$ uniformly at random.
  Contract edge $(u, v)$ to get $G'$ with new node $w$
  Set $S(w) \leftarrow S(u) \cup S(v)$. ▷ new super-node
  Update $G \leftarrow G'$.
**return** $S(v)$ for $v \in V$. ▷ cut with 2 nodes

## Contraction algorithm: example execution


Reference: Thore Husfeldt

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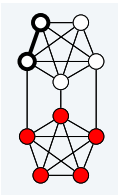
slide credit: Kevin Wayne / Pearson

## Contraction Algorithm Analysis

**Theorem.** Alg finds global min cut with probability at least $1/\binom{n}{2}$.

**Proof.** Suppose $(A, B)$ is a global min cut with $\text{cut}(A, B) = k$

- What could go wrong in the first step?
  - Select $(u, v)$ where $u \in A, v \in B$.



What is the probability of failure in the first step in this example?

A. 3/20
B. 3/23
C. 3/43
D. 1

## Contraction Algorithm Analysis

$$\Pr[\text{fail in round 1}] = \Pr[\text{select } u \in A, v \in B] = \frac{k}{|E|}$$

- To upper bound failure probability, need lower bound on $|E|$

Suppose $G$ is a graph with global minimum cut of size $k$. Then the degree of every node is always

A. At least $k$
B. Exactly $k$
C. At most $k$
D. At least $2m/n$

Every node has degree $\geq k \implies |E| \geq \frac{1}{2}kn$.

## Contraction Algorithm Analysis

$$\Pr[\text{fail in round 1}] = \Pr[\text{select } u \in A, v \in B] = \frac{k}{|E|}$$

- $\Pr[\text{fail in round 1}] \leq \frac{k}{\frac{1}{2}kn} = \frac{2}{n}$
- $p_{\text{suc}}(1) = \Pr[\text{success in round 1}] \geq 1 - \frac{2}{n} = \frac{n-2}{n}$
- Each time, $|E'| \geq kn'/2$, so success probability $\geq \frac{n'-2}{n'}$
- Success only if success in each round (product of probabilities)
- $p_{\text{suc}} = p_{\text{suc}}(1)p_{\text{suc}}(2)\cdots p_{\text{suc}}(n-2) \geq \frac{n-2}{n}\frac{n-3}{n-1}\cdots\frac{1}{3} = \frac{2}{n(n-1)}$

## Contraction Algorithm: Repeat for Success

**Theorem.** Alg finds global min cut with probability at least $1/\binom{n}{2}$.

**Corollary.** If run $\binom{n}{2} \ln n$ times, success probability is $\geq 1 - \frac{1}{n}$.

**Proof.**

$$\Pr[\text{Fail all } t \text{ times}] \leq \left(1 - \frac{1}{\binom{n}{2}}\right)^t$$

If $t = c\binom{n}{2}$ this is at most $e^{-c}$.

(From calculus: $(1 - 1/x)^x \leq 1/e = \lim_{x\to\infty}(1 - 1/x)^x$)

If run $\binom{n}{2} \ln n$ times, failure probability $\leq e^{-\ln n} = \frac{1}{n}$

## How many minimum cuts?

A graph may have several minimum cuts

When computing probability to succeed, we actually proved more!

We've shown the probability to return *any* minimum cut is $\geq \frac{2}{n(n-1)}$

But any two cuts are distinct! Let their number be $c$

$p_{\text{min-cut}} = p_{\text{min-cut}_1} + \ldots + p_{\text{min-cut}_c} \geq c\frac{2}{n(n-1)}$

But $p_{\text{min-cut}} \leq 1 \;\Rightarrow\; c \leq n(n-1)/2$

## Median Find

**Problem.** Given a set of numbers $S = \{a_1, \ldots, a_n\}$ the median is the number in the middle if the numbers were sorted.

- If $n$ is odd then $k$th smallest element where $k = (n+1)/2$.
- If $n$ is even then $k$th smallest element where $k = n/2$.

Deterministic algorithm?

- Sort numbers, take $k$th smallest.
- $O(n \log n)$.

## More Generally: $k^{\text{th}}$ Smallest

**Problem.** Given a set of numbers $S = \{a_1, \ldots, a_n\}$ and number $k$, return $k$th smallest number. (Assume no duplicates)

Special cases:

- $k = 1$: minimum element $O(n)$
- $k = n$: maximum element $O(n)$.

Why is it $O(n \log n)$ for $k = n/2$?

## Divide and Conquer Algorithm

- Choose splitter (or pivot) $a_i \in S$
- Form sets $S^- = \{a_j : a_j < a_i\}, S^+ = \{a_j : a_j > a_i\}$.

If:

- $|S^-| = k - 1$: $a_i$ is the target.
- $|S^-| \geq k$: recurse on $(S^-, k)$.
- $|S^-| < k - 1$, recurse on $(S^+, k - (|S^-| + 1))$.

Can do deterministic $O(n)$ algorithm along same idea
[Blum, Floyd, Pratt, Rivest, Tarjan '72]

Today: randomized

## Pseudocode

SELECT(S,k):
  Choose splitter $a_i \in S$.
  **for** each $a_j \in S$ **do**
    Put $a_j \in S^-$ if $a_j < a_i$.
    Put $a_j \in S^+$ if $a_j > a_i$.
  If $|S^-| = k - 1$, then return $a_i$.
  If $|S^-| \geq k$, return SELECT$(S^-, k)$.
  Else, return SELECT$(S^+, k - (|S^-| + 1))$.

Looks kind of like quicksort...

**Fact.** Algorithm is correct.

## How to choose splitter?

We want recursive calls to work on much smaller sets.

- Best case, splitter is the median:

$$T(n) \leq T(n/2) + cn \Rightarrow O(n) \text{ runtime}$$

- Worst case, splitter is largest element:

$$T(n) \leq T(n-1) + cn \Rightarrow O(n^2) \text{ runtime}$$

- Middle case, splitter separates $\epsilon n$ elements

$$T(n) \leq T((1-\epsilon)n) + cn$$
$$T(n) \leq cn\left[1 + (1-\epsilon) + (1-\epsilon)^2 + \ldots\right] \leq \frac{cn}{\epsilon}$$

How can we stay close to the best case?

## Randomized Splitters

**Idea.** Choose splitter uniformly at random.

**Analysis.** Phase $j$ when $n(3/4)^{j+1} \leq |S| \leq n(3/4)^j$.

- **Claim.** Expect to stay in phase $j$ for two rounds.
  - Call splitter *central* if separates $1/4$ fraction of elements.
  - $\Pr[\text{central splitter}] = 1/2$.
  - If $X$ is number of attempts until central splitter,

$$\begin{aligned}\mathbf{E}[X] &= \sum_{j=1}^{\infty} j \Pr[X = j] = \sum_{j=1}^{\infty} jp(1-p)^{j-1} \\ &= \frac{p}{1-p}\sum_{j=1}^{\infty} j(1-p)^j = \frac{p}{1-p}\frac{(1-p)}{p^2} \\ &= \frac{1}{p}\end{aligned}$$

## Analysis

- Let $Y$ be a r.v. equal to number of steps of the algorithm
- $Y = Y_0 + Y_1 + Y_2 + \ldots$ where $Y_j$ is steps in phase $j$
- One iteration in phase $j$ takes $cn(3/4)^j$ steps.
- $\mathbf{E}[Y_j] \leq 2cn(3/4)^j$ since we expect two iterations.

$$\begin{aligned}\mathbf{E}[Y] &= \sum_j \mathbf{E}[Y_j] \leq \sum_j 2cn(3/4)^j \\ &= 2cn\sum_j (3/4)^j \leq 8cn\end{aligned}$$

### Theorem

*Expected running time of* SELECT*(n,k) is* $O(n)$.

## Applications

- Randomized median find in expected linear time

Quicksort (Sketch)

- Choose pivot at random. Form $S^-, S^+$.
- Recursively sort both.
- Concatenate together.

**Theorem.** Quicksort has expected $O(n \log n)$ time.

## Review: Randomized Algorithms

- Efficient in expectation
- Optimal with high probability
- Break (undesired) symmetry
- Show some solution exists, or derive bound on number

Types of randomized algorithms:

- Fail with some small probability (Monte Carlo)
- Always succeed, but running time may be large (Las Vegas)