COMPSCI 311: Introduction to Algorithms

Lecture 17: Network Flow

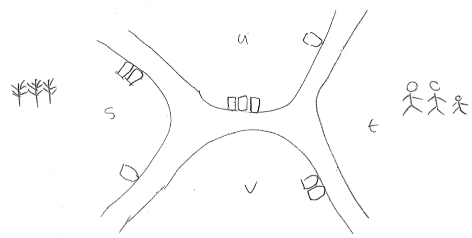
Marius Minea

University of Massachusetts Amherst

slides credit: Dan Sheldon

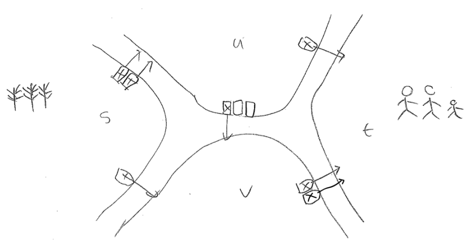
27 March 2019

A Puzzle

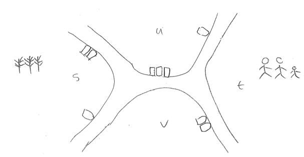


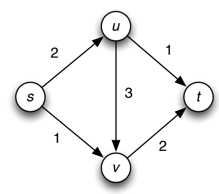
How many loads of grain can you ship from s to t?
Which boats are used?

A Puzzle



Max-Flow Problem: Flow Network

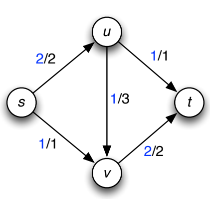




Problem input is a **flow network**

- Directed graph
- Source node s
- Target node or *sink* t
- Edge capacities $c(e) \geq 0$

Solution: A Flow



A **network flow** is an assignment of values $f(e)$ to each edge e, which satisfy:

- Capacity constraints: $0 \leq f(e) \leq c(e)$ for all e
- Flow conservation:

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

  for all $v \notin \{s, t\}$.
- **Max flow problem**: find a flow of maximum value
- Value $v(f)$ of flow f = total flow on edges leaving source

Network Flow

- Previous topics were design techniques (Greedy, Divide-and-Conquer, Dynamic Programming)
- Network flow: a specific class of problems with many applications

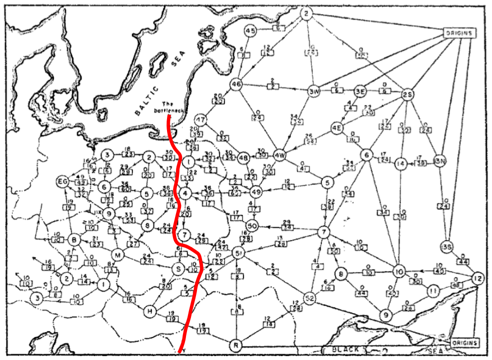
- Direct applications: commodities in networks
  - transporting goods on the rail network
  - packets on the internet
  - gas through pipes
- Indirect applications:
  - Matching in graphs
  - Airline scheduling
  - Baseball elimination

Plan: design and analyze algorithms for max-flow problem, then apply to solve other problems

First, a Story About Flow and Cuts

Key theme: flows in a network are intimately related to cuts

Soviet rail network (Harris & Ross, RAND report, 1955)



On the history of the transportation and maximum flow problems. Alexander Schrijver, Math Programming, 2002.

Clicker Question

Let's recall how a cut is defined:

A: A partition of graph vertices into two subsets

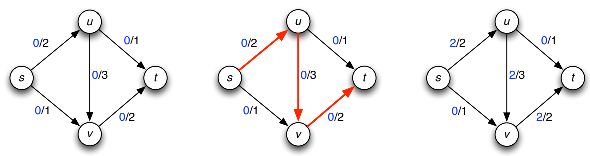
B: A partition of nodes so that the graph is bipartite

C: A set of edges that give a matching between two node sets

D: A set of edges between two node sets so that no two edges cross

Designing a Max-Flow Algorithm

First idea: initialize to zero flow, then repeatedly "augment" flow on paths from s to t until we can no longer do so.



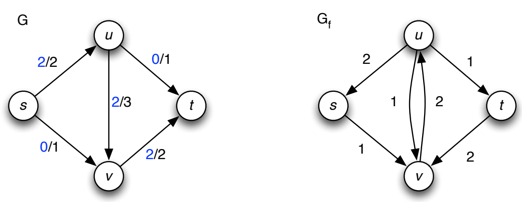
Problem: we are stuck, all paths from s to t have a *saturated* edge.

"In dealing with the usual railway networks a single flooding, followed by removal of bottlenecks, should lead to a maximal flow." (Boldyreff, RAND report, 1955)

We'd like to "augment" $s \xrightarrow{+1} v \xleftarrow{-1} u \xrightarrow{+1} t$, but this is not a real $s \to t$ path. How can we identify such an opportunity?

Residual Graph

The residual graph G_f identifies ways to increase flow on edges with leftover capacity, or decrease flow on edges already carrying flow:



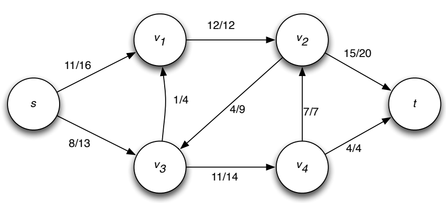
For each original edge $e = (u, v)$ in G, it has:

- A forward edge $e = (u, v)$ with *residual capacity* $c(e) - f(e)$
- A reverse edge $e' = (v, u)$ with *residual capacity* $f(e)$

Edges with zero residual capacity are omitted

Exercise: Residual Graph

G



Let G and f be as depicted above.
How many edges are there in the residual graph ?

A. 9

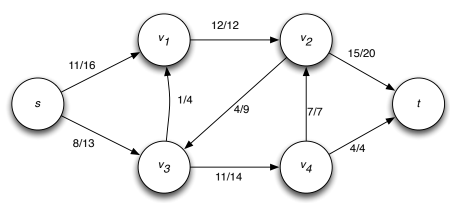
B. 12

C. 15

D. 18

Exercise: Residual Graph

G



Let G and f be as depicted above.
How many edges in the residual graph have capacity > 10?

A. 0

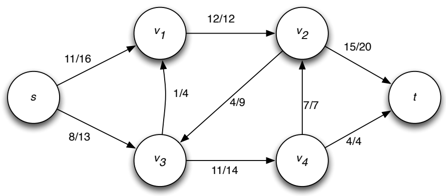
B. 1

C. 3

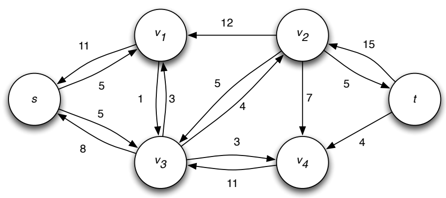
D. 4

Exercise: Residual Graph

G

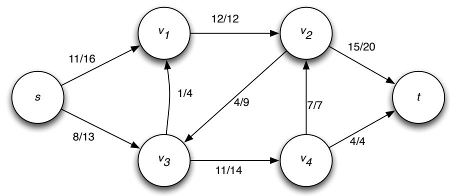


G_f

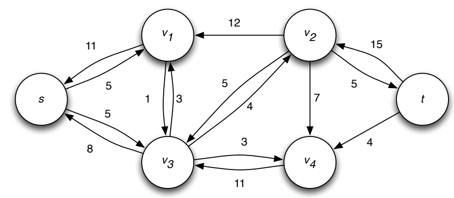


Exercise: Draw the Residual Graph

G

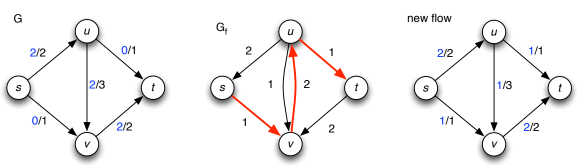


G_f



Augment Operation

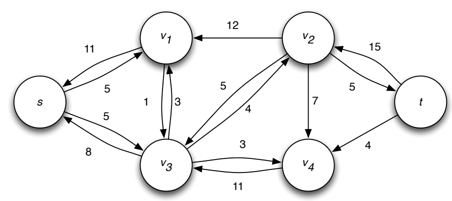
Revised Idea: use paths in the *residual* graph to augment flow



- $P = s \to v \to u \to t$ has bottleneck capacity 1.
- Increase flow for forward edges, decrease for backward edges.
- Augment $s \xrightarrow{+1} v \xleftarrow{-1} u \xrightarrow{+1} t$

Clicker Question

What is the highest bottleneck capacity of an augmenting path?



A. 1

B. 4

C. 5

D. 11

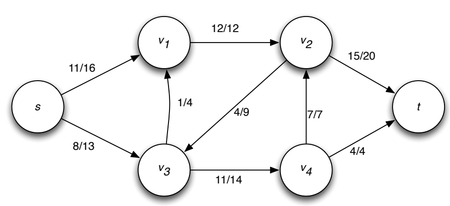
Augment Operation

Revised Idea: use paths in the *residual* graph to augment flow

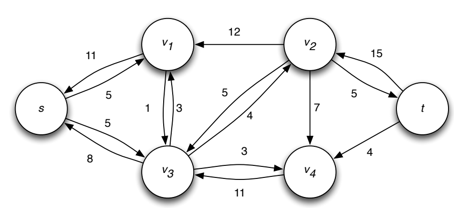
Augment(f, P)
 Let b = bottleneck(P, f) ▷ least residual capacity in P
 for each edge (u, v) in P **do**
  if (u, v) is a forward edge **then**
   Let $e = (u, v)$ be the original edge
   $f(e) = f(e) + b$ ▷ increase flow on forward edges
  else (u, v) is a backward edge
   Let $e = (v, u)$ be the original edge
   $f(e) = f(e) - b$ ▷ decrease flow on backward edges
  end if
 end for

Augment Example

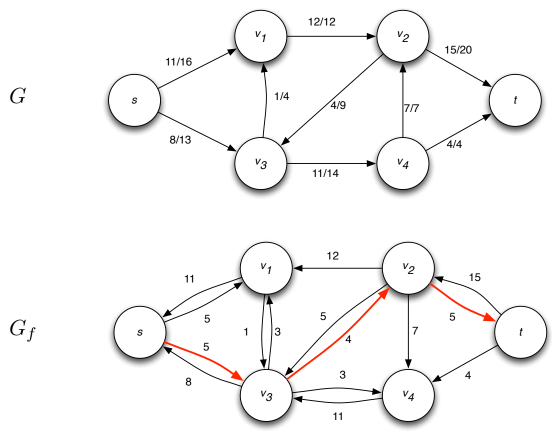
G



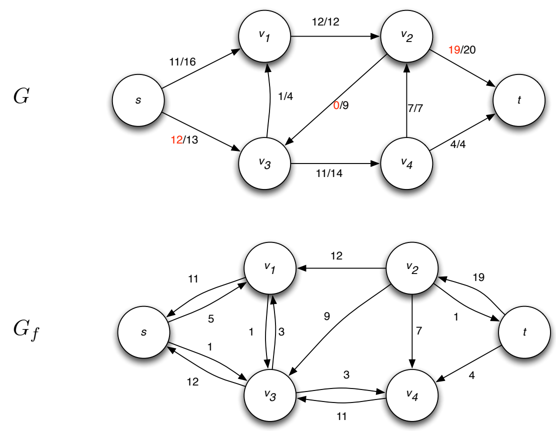
G_f



Augmenting Path



New Flow



Ford-Fulkerson Algorithm

Repeatedly find augmenting paths in the residual graph and use them to augment flow!

Ford-Fulkerson(G, s, t)
- ▷ Initially, no flow
- Initialize $f(e) = 0$ for all edges e
- Initialize $G_f = G$
- ▷ Augment flow as long as it is possible
- **while** there exists an s-t path P in G_f **do**
  - f = Augment(f, P)
  - update G_f
- **end while**
- return f

Clicker Question

Given a graph G and a flow f, how can you test if f is a maximum flow?

A. Check whether all edges from s are saturated.

B. Check whether all edges into t are saturated.

C. Check for an $s \to t$ path in the residual graph G_f.

D. Check for an $t \to s$ path in the residual graph G_f.

Ford-Fulkerson Analysis

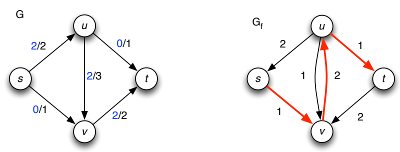
- Step 1: argue that F-F returns a flow
- Step 2: analyze termination and running time
- Step 3: argue that F-F returns a maximum flow

Step 1: F-F returns a flow

Claim: If f is a flow then f' = Augment(f, P) is also a flow.

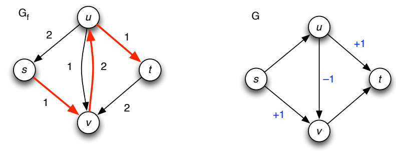
Proof idea. Verify two conditions for f' to be a flow: capacity and flow conservation.

Prove: Capacity



- Suppose original edge is $e = (u, v)$
- If e appears in P as a forward edge ($u \xrightarrow{+b} v$), then flow increases by bottleneck capacity b, which is at most $c(e) - f(e)$, so does not exceed $c(e)$
- If e appears in P as a reverse edge ($v \xleftarrow{-b} u$), then flow decreases by bottleneck capacity b, which is at most $f(e)$, so is at least 0

Prove: Flow Conservation



- Consider any node v in the augmenting path, and do a case analysis on the types of the incoming and outgoing edge:

residual graph: $P = s \rightsquigarrow u \longrightarrow v \longrightarrow w \rightsquigarrow t$

original graph:
$u \xrightarrow{+b} v \xrightarrow{+b} w$
$u \xrightarrow{+b} v \xleftarrow{-b} w$
$u \xleftarrow{-b} v \xrightarrow{+b} w$
$u \xleftarrow{-b} v \xleftarrow{-b} w$

- In all cases, the change in incoming flow to v is equal to the change in outgoing flow from v.

Step 2: Termination and Running Time

Assumption: All capacities are integers. By nature of F-F, all flow values and residual capacities remain integers during the algorithm.

Running time:

- In each F-F iteration, flow increases by at least 1. Therefore, number of iterations is at most $v(f^*)$, where f^* is the final flow.
- Let C_s be the total capacity of edges leaving source s.
- Then $v(f^*) \leq C_s$.
- So F-F terminates in at most C_s iterations

Running time per iteration? Cost of finding an augmenting path
How to find one? Any graph search: $O(m + n)$

Step 3: F-F returns a maximum flow

We will prove this by establishing a deep connection between flows and cuts in graphs: the max-flow min-cut theorem.

- An s-t cut (A, B) is a partition of the nodes into sets A and B where $s \in A$, $t \in B$
- Capacity of cut (A, B) equals

$$c(A, B) = \sum_{e \text{ from } A \text{ to } B} c(e)$$

- Flow across a cut (A, B) equals

$$f(A, B) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$