COMPSCI 311: Introduction to Algorithms

Lecture 13: Dynamic Programming

Marius Minea

University of Massachusetts Amherst

slides credit: Dan Sheldon

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Algorithm Design Techniques

- Greedy
- Divide and Conquer
- Dynamic Programming
- Network Flows

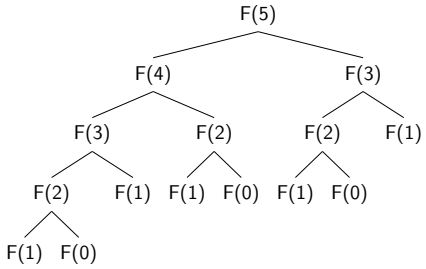
Learning Goals

	Greedy	Divide and Conquer	Dynamic Programming
Formulate problem			
Design algorithm		✓	✓
Prove correctness	✓		
Analyze running time		✓	
Specific algorithms	✓		Bellman-Ford shortest paths

Recursion May Be Easy .. But Sometimes Inefficient

Fibonacci sequence: F(0) = 0, F(1) = 1, F(n) = F(n-1) + F(n-2)

Compute by straightforward recursion:



Clicker Question 1

Consider the following function to compute Fibonacci numbers:

```
fib(n):
  if n < 2 return n;
  return fib(n-1) + fib(n-2);
```

The complexity of fib(n) is

A. $\Theta(n^{\log_2 3})$

B. $\Theta(F(n))$

C. $\Theta(2^n)$

D. $\Theta(n!)$

Dynamic Programming Recipe

- Step 1: Devise simple recursive algorithm
  - Usually for optimization: try all choices at one level, solve subproblems
  - But: subproblems often shared ⇒ redundant computation (may be exponential time)
- Step 2: Write recurrence for optimal value
- Step 3: Design iterative algorithm (avoids redundancy)

Weighted Interval Scheduling

- TV scheduling problem: Given n shows with start time s_i and finish time f_i, watch as many shows as possible, with no overlap.
- A Twist: Each show has a value v_i. We want a set of shows S, with no overlap and maximum value $\sum_{i \in S} v_i$.
- Greedy? It worked for case without values
- Problem formulation
  - Show (job) j has value v_j, start time s_j, finish time f_j
  - Assume shows sorted by finishing time $f_1 \leq f_2 \leq \ldots \leq f_n$
  - Shows i and j are compatible if they don't overlap
  - **Goal**: subset of non-overlapping jobs with maximum value

Step 1: Recursive Algorithm

- **Observation**: Let O be the optimal solution.
  Last interval n is either in O or it isn't.
  In both cases, we get a *smaller instance* of the same problem.
- Recursive algorithm: value of optimal subset of first j shows (going backwards from j)

Compute-Value(j)
  Base case: if $j = 0$ return 0
  Case 1: $j \in O$
  Let $i < j$ be highest-numbered show compatible with j
  val1 $= v_j +$ Compute-Value(i)
  Case 2: $j \notin O$
  val2 = Compute-Value($j - 1$)
  return max(val1, val2)

Clicker Question 2

Compute-Value(j)
  if $j = 0$ return 0
  Let $i < j$ be highest-numbered show compatible with j
  val1 $= v_j +$ Compute-Value(i)
  val2 = Compute-Value($j - 1$)
  return max(val1, val2)

The running time of this recursive solution is

A: $O(n \log n)$

B: $O(n^2)$

C: $O(1.618^n)$

D: $O(2^n)$

Running Time?

- Recursion tree
- $\approx 2^n$ subproblems $\Rightarrow$ exponential time
- Only n *unique* subproblems.
  Save work by ordering computation to solve each problem once.

Step 2: Recurrence

- Recurrence: directly expresses solution (optimal value) in terms of solutions for subproblems (recursive structure)
- OPT(j) = value of optimal solution on first j shows
- p_j: highest-numbered show that is compatible with j
- Recurrence:

$$\text{OPT}(0) = 0$$
$$\text{OPT}(j) = \max\{\underbrace{v_j + \text{OPT}(p_j)}_{\text{Case 1}}, \underbrace{\text{OPT}(j-1)}_{\text{Case 2}}\}$$

Recursive Algorithm vs. Recurrence

- Compute-Value(j)
  If $j = 0$ return 0
  val1 $= v_j +$ Compute-Value(p_j)
  val2 = Compute-Value($j - 1$)
  return max(val1, val2)
- Recurrence

$$\text{OPT}(j) = \max\{v_j + \text{OPT}(p_j), \text{OPT}(j-1)\}$$
$$\text{OPT}(0) = 0$$

- Direct correspondence between the algorithm and recurrence
  - **Tip**: start by writing the recursive algorithm and translating it to a recurrence (replace method name by "OPT")
  - After some practice, skip straight to the recurrence

Step 3: Iterative "Bottom-Up" Algorithm

WeightedIS
  Initialize array M of size n to hold optimal values
  $M[0] = 0$ ▷ Value of empty set
  for $j = 1$ to n **do**
    $M[j] = \max(v_j + M[p_j], M[j-1])$
  end for

Usually we directly convert recurrence to appropriate `for` loop.

Pay attention to dependence on previously-computed array entries to know in which direction to iterate.

Memoization

Intermediate approach: keep recursive function structure, but store value in array on first computation, and reuse it

  Initialize array M of size n to *empty*, M[0] = 0
  function Mfun(j)
  if M[j] = *empty*
    M[j] = max(v_j + Mfun(p_j), Mfun(j-1))
  return M[j]

Can help if we have recursive structure but unsure of iteration order
Or as intermediate step in converting to iteration

Clicker Question 3

The asymptotic complexity of the memoized algorithm is

A: Same as the initial recurrence

B: Between the initial recurrence and the iterated version

C: Same as the iterated version

Epilogue: Recovering the Solution (1)

Idea: modify the algorithm to what choice is made at each iteration

WeigthedIS
  Initialize array $M[0 \dots n]$ to hold optimal values
  Initialize array choose$[1 \dots n]$ to hold choices
  $M[0] = 0$
  for $j = 1$ to n **do**
    $M[j] = \max(v_j + M[p_j], M[j-1])$
    Set choose$[j] = 1$ if first value is bigger, and 0 otherwise
  end for

Epilogue: Recovering the Solution (2)

Then trace back from end and "execute" the choices
  Use algorithm above to fill in M and choose arrays
  $O = \{\}$
  $j = n$
  while $j > 0$ **do**
    if choose$(j) == 1$ **then**
      $O = O \cup \{j\}$
      $j = p_j$
    else
      $j = j - 1$
    end if
  end while

- Tip: first do algorithm to just compute optimal value; then modify it to compute the actual solution

Review

- Recursive algorithm → recurrence → iterative algorithm
- Three ways of expressing value of optimal solution for smaller problems
  - Compute-Value(j). Recursive algorithm—arguments identify subproblems.
  - OPT(j). Mathematical expression. Write a recurrence for this that matches recursive algorithm.
  - $M[j]$. Array to hold optimal values. Entries filled in during iterative algorithm.

Key Step: Identify Subproblems

- Finding solution means: make "first choice", then recursively solve a smaller instance of same problem.
- First example: Weighted Interval Scheduling
  - **Binary** first choice: $j \in O$ or $j \notin O$?
- Next example: rod cutting
  - First choice has n **options**

Rod Cutting

- **Problem Input**:
  - Steel rod of length n, can be cut into integer lengths
  - Price based on length, $p(i)$ for a rod of length i
- **Goal**
  - Cut rods into lengths $i_1, \ldots, i_k$ such that $i_1 + i_2 + \ldots i_k = n$.
  - Maximize value $p(i_1) + p(i_2) + \ldots + p(i_n)$

First choice?

- Greedy? Cut length with maximum price
  Or: cut piece with maximum price *per length* ?
- Divide and Conquer:
  Break rod at some (integer) point. Recurse for pieces.
- Dynamic Programming:
  Choose length i of first piece, then recurse on rest

Steps 1 and 2

Step 1: Recursive Algorithm.

CutRod(j)
  if $j = 0$ **then** return 0
  $v = 0$
  for $i = 1$ to j **do**
    $v = \max(v, p[i] + \text{CutRod}(j - i))$
  end for
  return v

- Running time for CutRod(n)? $\Theta(2^n)$

Step 2: Recurrence

$$\text{OPT}(j) = \max_{1 \le i \le j} \{p_i + \text{OPT}(j - i)\}$$
$$\text{OPT}(0) = 0$$

Step 3: Iterative Algorithm

- Array $M[0..n]$ where $M[i]$ holds value of $\text{OPT}(i)$.
  Order to fill M? From 0 to n.

  CutRod-Iterative
    Initialize array $M[0..n]$
    Set $M[0] = 0$
    for $j = 1$ to n **do**
      $v = 0$
      for $i = 1$ to j **do**
        $v = \max(v, p[i] + M[j - i])$
      end for
      Set $M[j] = v$
    end for

- Running time? $\Theta(n^2)$ Note: body of for loop identical to recursive algorithm, directly implements recurrence

Epilogue: Recover Optimal Solution

Run previous algorithm to fill in M array
cuts $= \{\}$
$j = n$
while $j > 0$ **do**
  $i^* =$ null, $v = 0$ ▷ i^* is the selected cut, v is its value
  for $i = 1$ to j **do**
    if $p[i] + M[j - i] > v$ **then**
      $i^* = i$
      $v = p[i] + M[i]$
    end if
  end for
  $j = j - i^*$
  cuts $=$ cuts $\cup \{i^*\}$
end while