COMPSCI 311: Introduction to Algorithms

Lecture 6: Greedy Algorithms – Exchange Arguments

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Algorithm Design—Greedy

Greedy: make a single "greedy" choice at a time, don't look back.

	Greedy
Formulate problem	?
Design algorithm	easy
Prove correctness	hard
Analyze running time	easy

Focus is on proof techniques

- Last time: "greedy stays ahead" (inductive proof)
- This time: exchange argument

Scheduling to Minimize Lateness

- You have a very busy month: n assignments are due, with different deadlines

```
Assignments:
    1: |---|              (len=1, due=2)
    2: |---|---|          (len=2, due=5)
    3: |---|---|---|      (len=3, due=6)
    4: |---|---|          (len=2, due=7)

Deadlines:
            d1          d2  d3  d4
    |---|---|---|---|---|---|---|---|---|
    0   1   2   3   4   5   6   7   8   9
```

- How should you schedule your time to "minimize lateness"?

Scheduling to Minimize Lateness

Let's formalize the problem. The input is:

- t_j = length (in days) to complete assignment j (or "job" j)
- d_j = deadline for assignment j

What does a schedule look like?

- s_j = start time for assignment j (selected by algorithm)
- $f_j = s_j + t_j$ finish time

How to evaluate a schedule?

- Lateness of assignment j is $\ell_j = \begin{cases} 0 & \text{if } f_j \leq d_j \\ f_j - d_j & \text{if } f_j > d_j \end{cases}$
- Maximum lateness $L = \max_j \ell_j$

Goal: schedule so maximum lateness is as small as possible

Clicker Question 1

An algorithm to minimize maximum lateness will also find a schedule that is not late, if that is possible ?

- A) Yes
- B) No, because the lateness function is not linear
- C) No, because it minimizes the maximum lateness, whereas we want all jobs to have lateness zero

Possible Greedy Approaches

- **Note**: scheduling work back-to-back (no idle time) can't hurt
  $\Rightarrow$ schedule determined just by order of assignments

```
    1: |---|              (len=1, due=2)
    2: |---|---|          (len=2, due=5)
    3: |---|---|---|      (len=3, due=6)
    4: |---|---|          (len=2, due=7)
```

- What order should we choose?
  - *Shortest Length*: ascending order of t_j.
  - *Smallest Slack*: ascending order of $d_j - t_j$.
  - *Earliest Deadline*: ascending order of d_j.
- *Earliest deadline first* is better in example.
  Let's prove it is always optimal.

Clicker Question 2

If two jobs have the same deadline, the earliest deadline first algorithm should schedule

- A) The shortest job first, because that has a higher chance of finishing before the deadline
- B) The longest job first, because then its lateness will be minimized
- C) Does not matter

Identical Maximum Lateness

Claim: If in a schedule we swap two jobs with the same deadline, we get the same maximum lateness. True?

Not necessarily, if the jobs are not in earliest deadline first order! (Example)

Claim: If in an EDF schedule, we swap two jobs with the same deadline, we get the same maximum lateness.

Proof: Since the schedules are EDF, all jobs with the same deadline are scheduled in a consecutive block.
Then among those, the last one has the maximum lateness.
That finishing time does not change by swapping schedules within the block.

Corollary All EDF schedules have the same maximum lateness.

Exchange Argument (False Start)

Assume jobs ordered by deadline $d_1 \leq d_2 \leq ... \leq d_n$, so the greedy ordering is simply

$$A = 1, 2, \ldots, n$$

Claim: A is optimal

Proof attempt: Suppose for contradiction that A is not optimal. Then, there is an optimal solution O with $O \neq A$

- Since $O \neq A$, there must be two jobs i and j that are out of order in O (e.g. $O = 1, 3, 2, 4$)
- Let's swap i and j and show we get a *better* solution O'
- $\implies O$ is *not* optimal. Contradiction, so A must be optimal.

Problem? O' may still be optimal. Example?

Can't do proof by contradiction in this way.

Exchange Argument (Correct)

Suppose O optimal and $O \neq A$. Then we can modify O to get a new solution O' that is:

1. No worse than O
2. Closer to A is some measurable way

$$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \ldots \rightarrow A(\text{optimal})$$

High-level idea: gradually transform O into A without hurting solution, thus preserving optimality.
Concretely: show 1 and 2 above.

Exchange Argument for Scheduling to Minimize Lateness

Recall $A = 1, 2, \ldots, n$. For $S \neq A$, say there is an inversion if i comes before j but $j < i$ (thus $d_j \leq d_i$)
Claim: if S has an inversion, S has a consecutive inversion—one where i comes immediately before j. Why?

Main result: let $O \neq A$ be an optimal schedule. Then O has a consecutive inversion i, j. We can swap i and j to get a new schedule O' such that:

1. O' has one less inversion than O
2. Maximum lateness of O' is at most maximum lateness of O

Proof:

1. Obvious
2. Next slide(s)

Proof (Lateness does not increase)

Swapping a consecutive inversion (i precedes j; $d_j \leq d_i$)

```
                |-------i-------|---j---|   O

     dj   di    |---j---|-------i-------|   O'
-----|-----|--------------------------------------->
```

Consider the lateness ℓ'_k of each job k in O':

- If $k \notin \{i, j\}$, then lateness is unchanged: $\ell'_k = \ell_k$
- Job j finishes earlier in O' than O: $\ell'_j \leq \ell_j$
- Finish time of i in O' = finish time of j in O. Therefore

$$\ell'_i = f'_i - d_i = f_j - d_i \leq f_j - d_j = \ell_j$$

Conclusion: $\max_k \ell'_k \leq \max_k \ell_k$. Therefore O' is still optimal.

Wrap-Up

For any optimal $O \neq A$ we showed that we could transform O to O' such that:

1. O' is still optimal
2. O' has one less inversion than A

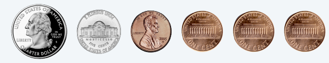
$$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \ldots \rightarrow A(\text{optimal})$$

Since there are at most $\binom{n}{2}$ inversions, by repeating the process a finite number of times we see that A is optimal.

Coin changing

Goal. Given U. S. currency denominations { 1, 5, 10, 25, 100 }, devise a method to pay amount to customer using fewest coins.

Ex. 34¢.



Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex. $2.89.

Greedy algorithms I: quiz 1

Is the cashier's algorithm optimal?

A. Yes, greedy algorithms are always optimal.

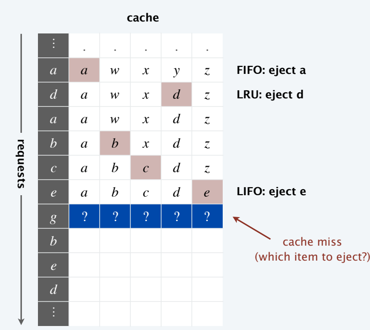
B. Yes, for any set of coin denominations $c_1 < c_2 < \ldots < c_n$ provided $c_1 = 1$.

C. Yes, because of special properties of U.S. coin denominations.

D. No.

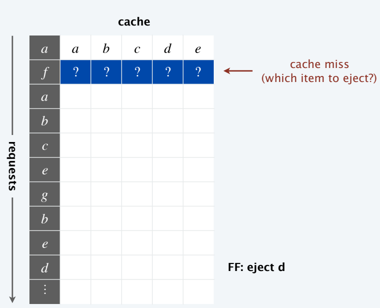
Optimal offline caching: greedy algorithms

LIFO/FIFO. Evict item brought in least (most) recently.
LRU. Evict item whose most recent access was earliest.
LFU. Evict item that was least frequently requested.



Optimal offline caching: farthest-in-future (clairvoyant algorithm)

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.



Theorem. [Bélády 1966] FF is optimal eviction schedule.
Pf. Algorithm and theorem are intuitive; proof is subtle.

Wrap-Up: Greedy Algorithms

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Proof techniques

- Last time: "greedy stays ahead" (inductive proof)
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Need to formulate precisely; careful not to get arguments wrong!