COMPSCI 311: Introduction to Algorithms

Lecture 5: Greedy Algorithms

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Greedy Algorithms

We are moving on to our study of algorithm design techniques:

- Greedy
- Divide-and-conquer
- Dynamic programming
- Network flow

Get a sense of "greedy" algorithms, then characterize them

Interval Scheduling

- In the 80s, your only opportunity to watch a specific TV show was the time it was broadcast. Unfortunately, on a given night there might be multiple shows that you want to watch and some of the broadcast times overlap.

  Example: [1, 4], [2, 3], [2, 7], [4, 7], [3, 6], [6, 10], [5, 7]

- You want to watch the highest number of shows. Which subset of shows do you pick?
- Fine print: assume you like all shows equally, you only have one TV, and you need to watch shows in their entirety.

Formalizing Interval Scheduling

Let's formalize the problem

- Shows $1, 2, \ldots, n$
  (more generally: "requests" to be fulfilled with a given resource)
- s_j: start time of show j
- f_j, also written $f(j)$: finish time of show j
- Shows i and j are compatible if they don't overlap.
- Set A of shows is compatible all pairs in A are compatible.
- Set A of shows is optimal... if it is compatible and no other compatible set is larger.

Greedy Algorithms

- Main idea in greedy algorithms is to make one choice at a time in a "greedy" fashion.
  (Choose the thing that looks best, never look back...)
- We will sort shows in some "natural order" and choose shows one by one if they're compatible with the shows already chosen. Concretely:

$R \leftarrow$ set of all shows **sorted by some property**
$A \leftarrow \{\}$ ▷ selected shows
while R is not empty **do**
  Take first show i from R
  Add i to A
  Delete i and all overlapping shows from R
end while

Clicker Question 1

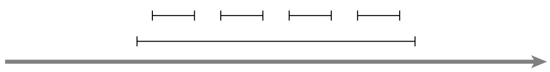
$R \leftarrow$ set of all shows **sorted by some property**
$A \leftarrow \{\}$ ▷ selected shows
while R is not empty **do**
  Take first show i from R
  Add i to A
  Delete i and all overlapping shows from R
end while

Because the given algorithm includes sorting, we can deduce it is

- A: $O(n \log n)$
- B: $\Omega(n \log n)$
- C: $\Theta(n \log n)$
- D: None of the above

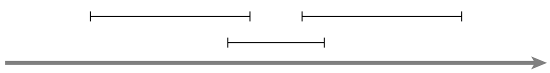
What's a "natural order" ?

- *Start Time*: Consider shows in ascending order of s_j.



(a)

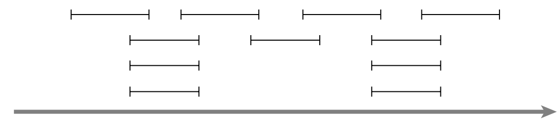
- *Shortest Time*: Consider shows in ascending order of $f_j - s_j$.



(b)

What's a "natural order" ?

- *Fewest Conflicts*: Let c_j be number of shows which overlap with show j. Consider shows in ascending order of c_j.



(c)

- *Finish Time*: Consider shows in ascending order of f_j.

We'll show that this works!

Analysis

Sorting shows by finish time gives an optimal solution in examples. Let's try to prove that it will always be optimal.

Let A be the set of shows returned by the algorithm when shows are sorted by finish time. What do we need to prove?

- A is compatible (obvious property of algorithm)
- A is optimal

We will prove A is optimal by a "greedy stays ahead" argument

Ordering by Finish Time is Optimal: "Greedy Stays Ahead"

- Let $A = i_1, \ldots, i_k$ be the intervals selected by the greedy algorithm
- Let $O = j_1, \ldots, j_m$ be the intervals of some optimal solution O
- Assume both are sorted by finish time

```
A: |--i1--||---i2---| ... |---ik---|
O: |---j1---||---j2---|     ...      |----jm----|
```

- Could it be the case that $m > k$?
- Observation: $f(i_1) \leq f(j_1)$. The first show in A finishes no later than the first show in O.
- **Claim** ("greedy stays ahead"): $f(i_r) \leq f(j_r)$ for all $r = 1, 2, \ldots$. The rth show in A finishes no later than the rth show in O.

"Greedy Stays Ahead"

- **Claim**: $f(i_r) \leq f(j_r)$ for all $r = 1, 2, \ldots$
- **Proof** by induction on r
- **Base case** ($r = 1$): i_r is the first choice of the greedy algorithm, which has the earliest overall finish time, so $f(i_r) \leq f(j_r)$

Induction Step

- Assume inductively that $f(i_{r-1}) \leq f(j_{r-1})$
- Assume for sake of contradiction that $f(i_r) > f(j_r)$

```
A: |--i1--| ... |---i(r-1)---||-------ir------|
O: |---j1---| ... |---j(r-1)---||----jr-----|
```

- But it must be the case that j_r is compatible with the first $r - 1$ shows in A, because (using induction hypothesis)

$$s(j_r) \geq f(j_{r-1}) \geq f(i_{r-1})$$

- Therefore, the greedy algorithm could have selected j_r instead of i_r. But j_r finishes sooner than i_r, which contradicts the algorithm.
- Therefore, it must be the case that $f(i_r) \leq f(j_r)$

Running Time?

$R \leftarrow$ set of all shows **sorted by finishing time**
$A \leftarrow \{\}$ ▷ selected shows
while R is not empty **do**
  Take first show i from R
  Add i to A
  Delete i and all overlapping shows from R
end while

Can we make loop better than n^2? Check conflict when *selecting* i

$R \leftarrow$ set of all shows **sorted by finishing time**
$A \leftarrow \{\}$, $end = 0$ ▷ last scheduled time
for show i from 1 to n **do**
  if $s_i \geq end$ **then**
    Add i to A; $end = f_i$ ▷ $O(1)$ iteration
  end if
end for

$\Theta(n \log n)$ — dominated by sort
Change in abstract version of algorithm makes it more efficient!

Algorithm Design—Greedy

Greedy: make a single "greedy" choice at a time, don't look back.

	Greedy
Formulate problem	?
Design algorithm	easy
Prove correctness	hard
Analyze running time	easy

Focus is on proof techniques. Next time: another proof technique.

Problem 2: Interval Partitioning

- Suppose you are in charge of UMass classrooms.
- There are n classes to be scheduled on a Monday where class j starts at time s_j and finishes at time f_j
- Your goal is to schedule *all* the classes such that the minimum number of classrooms get used throughout the day. Obviously two classes that overlap can't use the same room.

Possible Greedy Approaches

- Suppose the available classrooms are numbered $1, 2, 3, \ldots$
- We could run a greedy algorithm… consider the lectures in some natural order, and assign the lecture to the classroom with the smallest numbered that is available.
- What's a "natural order"?
  - *Start Time*: Consider classes in ascending order of s_j.
  - *Finish Time*: Consider classes in ascending order of f_j.
  - *Shortest Time*: Consider classes in ascending order of $f_j - s_j$.
  - *Fewest Conflicts*: Let c_j be number of classes which overlap with show j. Consider shows in ascending order of c_j.

How Many Classrooms Are Needed?

- Consider all points on the timeline
- Count how many classes run at that time
- Maximum number is called the *depth* of the set of intervals
- It's a lower bound on number of rooms needed (why?)
- Is this number sufficient ?

Interval Partitioning Algorithm

Sort the intervals by starting time
for $j = 1$ to n **do**
  for each $i < j$ overlapping interval j **do**
    exclude label of I_i for scheduling I_j
  end for
  if there is some nonexcluded label in 1 .. d **then**
    label I_j with that label
  end if
end for

Clicker Question 2

If the class with the next starting time is compatible with several rooms, it shoud be scheduled

- A) In the room with the earliest finishing time
- B) In the room with the latest finishing time
- C) In a room where nothing was scheduled so far
- D) Does not matter

Correctness of Interval Partitioning

Claim: Every resource will be assigned a label.

Proof? By contradiction, otherwise would have higher depth.

Claim: No two resources are assigned the same label.

Proof? Assume two intervals overlap, I_1 starting before I_2
Label of I_1 is excluded when scheduling I_2

Complexity of Interval Partitioning

Sort the intervals by starting time
for $j = 1$ to n **do**
  for each $i < j$ overlapping interval j **do**
    exclude label of I_i for scheduling I_j
  end for
  if some label in 1 .. d is not excluded **then**
    label I_j with that label
  end if
end for

Naive: $O(n \log n + n^2)$

Better: $O(n \log n + nd)$ (keep finishing time for each label)

Improved:

- keep a priority queue of last finishing times for each label
- find $\min$ in $O(1)$, update in $O(\log d)$
- outer loop becomes $O(n \log d)$

Clicker Question 3

An $O(n \log n + n \log d)$ implementation for interval partitioning is also

- A) $O(n \log n)$
- B) $O(n \log(nd))$
- C) Both A and B
- D) None of A or B