COMPSCI 311 Introduction to Algorithms

Lecture 4: Graphs

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Last time: Generic Graph Traversal

Let A = data structure of discovered nodes

Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A ▷ w is discovered
      end for
    end if
  end while

BFS: A is a queue (FIFO) DFS: A is a stack (LIFO)

Clicker Question 1

Put s in A
while A is not empty **do**
  Take a node v from A
  if v is not marked "explored" **then**
    Mark v as "explored"
    for each edge (v, w) incident to v **do**
      Put w in A ▷ w is discovered
    end for
  end if
end while

What is the maximum number of times a node w can be put in A?

- A: once
- B: $degree(w)$ times
- C: $2 \cdot degree(w)$ times
- D: $|V|$ times

BFS Implementation

Let A = empty Queue structure of discovered nodes

Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A ▷ w is discovered
      end for
    end if
  end while

Is this actually BFS? Yes.

BFS Running Time

How many times does each line execute?

Traverse(s)
  Put s in A 1
  while A is not empty **do** $2m$
    Take a node v from A $2m$
    if v is not marked "explored" **then** $2m$
      Mark v as "explored" n
      for each edge (v, w) incident to v **do** $2m$
        Put w in A $2m$
      end for
    end if
  end while

Running time $O(m + n)$

DFS Implementation

Let A = empty Stack structure of discovered nodes

Traverse(s)
  Put s in A
  while A is not empty **do**
    Take a node v from A
    if v is not marked "explored" **then**
      Mark v as "explored"
      for each edge (v, w) incident to v **do**
        Put w in A ▷ w is discovered
      end for
    end if
  end while

Is this actually DFS? Yes (reverse order for node neighbors)
Running time? $O(m + n)$

Clicker Question 2

```
DFS(u)
  Mark u as "explored"
  for each edge (u,v) do
    if v is not "explored" then
      Call DFS(v) recursively
    end if
  end for
```

```
Put s in A
while A is not empty do
  Take a node v from A
  if v is not "explored" then
    Mark v as "explored"
    for each edge (v,w) do
      Put w in A
    end for
  end if
end while
```

Suppose we have a tree with n nodes, height h and degree d.

Compare the memory used by recursive and non-recursive DFS (clarification: for the stack)

- A: recursive: $\Theta(hd)$, non-recursive: $\Theta(h)$
- B: recursive: $\Theta(h)$, non-recursive: $\Theta(hd)$
- C: recursive: $\Theta(n)$, non-recursive: $\Theta(hd)$
- D: recursive: $\Theta(h)$, non-recursive: $\Theta(d)$

Back to Connected Components

```
while There is some unexplored node s do
  BFS(s)
  Extract connected component containing s
end while
```

Running time?

Naive: $O(m+n)$ for each component
$\Rightarrow O(c(m+n))$ if c components.

Better: BFS on component C only works on nodes/edges in C

- Time for component C: $O(\#\text{edges in } C + \#\text{nodes in } C)$
- Total time: $O(m+n)$

Review and Outlook

- Graph traversal by BFS/DFS
  - Different versions of general exploration strategy
  - $O(m+n)$ time
  - Produce trees with useful properties (for other problems)
  - Basic algorithmic primitive — used in many other algorithms
    path from s to t, connected components
- Bipartite testing
- Directed graphs
  - Traversal
  - Strong connectivity
  - Topological sorting

Bipartite Graphs

Definition Graph $G = (V, E)$ is bipartite if V can be partitioned into sets X, Y such that every edge has one end in X and one in Y.

Can color nodes red/blue s.t. no edges between nodes of same color.

Examples

- Bipartite: student-college graph in stable matching
- Bipartite: client-server connections
- Not bipartite: "odd cycle" (cycle with odd # of nodes)
- Not bipartite: any graph containing odd cycle

Claim (easy): If G contains an odd cycle, it is not bipartite.

Bipartite Testing

Question Given $G = (V, E)$, is G bipartite?

Algorithm? **Idea**: run BFS from any node s

- L_0 = red
- L_1 = blue
- L_2 = red
- ...
- Even layers red, odd layers blue

What could go wrong? Edge between two nodes at same layer.

Algorithm

```
Run BFS from any node s
if there is an edge between two nodes in same layer then
  Output "not bipartite"
else
  X = even layers
  Y = odd layers
end if
```

Correctness? Recall: all edges between same or adjacent layers.

1. If there are no edges between nodes in the same layer, then G is bipartite.
2. If there is an edge between two nodes in the same layer, G has an odd cycle and is not bipartite. Proof?

Proof

- Let T be BFS tree of G and suppose (x, y) is an edge between two nodes in the layer j
- Let $z \in L_i$ be the least common ancestor of x and y
  (Useful in proofs: take least/greatest item with some property)
  - P_{zx} = path from z to x in T
  - P_{yz} = path from z to y in T
  - Path that follows P_{zx} then edge (x, y) then P_{yz} is a cycle of length $2(j - i) + 1$, which is odd
- Therefore G is not bipartite.

Directed Graphs

$G = (V, E)$

- $(u, v) \in E$ is a *directed* edge
- u points to v

Examples

- Facebook: undirected
- Twitter: directed
- Web: directed
- Road network: directed (if one-way roads)

Directed Graph Traversal

Reachability. Find all nodes reachable from some node s.

s-t shortest path.
What is the length of the shortest directed path from s to t?

Algorithm? BFS naturally extends to directed graphs.
Add v to L_{i+1} if there is a *directed* edge from L_i and v is not already discovered.

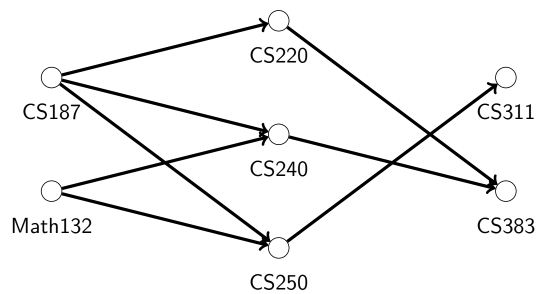
Some problems require us to consider the graph G^{rev} with edges reversed.

Useful to keep adjacency lists for both outgoing and incoming edges.

Directed Acyclic Graphs

Definition
A directed acyclic graph (DAG) is a directed graph with no cycles.

Models *dependencies*, e.g. course prerequisites:



Math: (strict) partial order (irreflexive, antisymmetric, transitive)

Topological Sorting

Definition A topological ordering of a directed graph is an ordering of the nodes such that all edges go "forward" in the ordering

- Label nodes $v_1, v_2, \ldots, v_n$ such that
- For all edges (v_i, v_j) we have $i < j$
- A way to order the classes so all prerequisites are satisfied

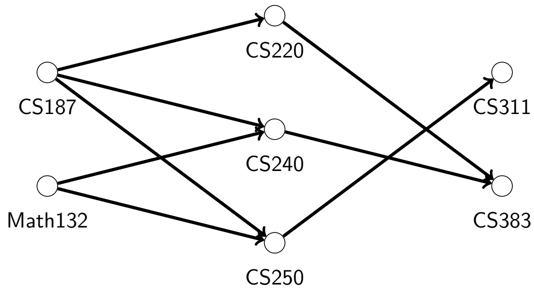
Q: Is a topological ordering possible for any graph?

Clicker Question 3

The maximum number of edges in a DAG with n nodes is

- A) $2(n - 1)$
- B) $2n - 1$
- C) $n(n - 1)/2$
- D) $n(n - 1)$

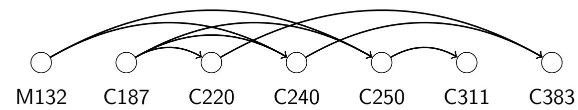
Topological Sorting



Exercise

1. Find a topological ordering.
2. Devise an algorithm to find a topological ordering.

Topological Ordering



Claim If G has a topological ordering, then G is a DAG.

Topological Sorting

Problem Given DAG G, compute a topological ordering for G.

topo-sort(G)
  while there are nodes remaining **do**
    Find a node v with no incoming edges
    Place v next in the order
    Delete v and all of its outgoing edges from G
  end while

Running time? $O(n^2 + m)$ easy. $O(m + n)$ more clever

Topological Sorting Analysis

- In a DAG, there is always a node v with no incoming edges. Try to prove. (contradiction, pigeonhole principle)
- Removing a node v from a DAG, produces a new DAG.
- Any node with no incoming edges can be first in topological ordering.

Theorem: G is a DAG if and only if G has a topological ordering.

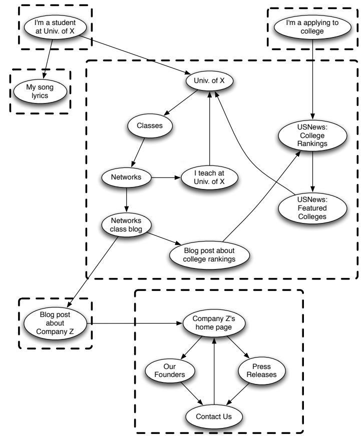
Topological Sorting in $O(m + n)$

topo-sort(G)
  while there are nodes remaining **do**
    Find a node v with no incoming edges
    Place v next in the order
    Delete v and all of its outgoing edges from G
  end while

Optimization: don't search every time for nodes w/o incoming edges

- Keep and update incoming edge count for each node (setup in $O(m + n)$, each update constant-time)
- Keep set of nodes of nodes with incoming edges; add node when its count becomes zero
- Running time: $O(m + n)$

Directed Graph Connectivity



Strongly connected graph. Directed path between any two nodes.

Strongly connected component (SCC). Maximal subset of nodes with directed path between any two.

SCCs can be found in time $O(m + n)$. (Tarjan, 1972)

Graph of SCCs (one node for each) is a DAG.