COMPSCI 311 Introduction to Algorithms

Lecture 3: Graphs

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slides credit: Dan Sheldon, Akshay Krishnamurthy, Andrew McGregor

Review: Asymptotics

Property	Definition / terminology
$f(n)$ is $O(g(n))$	$\exists c, n_0$ s.t. $f(n) \le cg(n)$ for all $n \ge n_0$ g is an asymptotic upper bound on f
$f(n)$ is $\Omega(g(n))$	$\exists c, n_0$ s.t. $f(n) \ge cg(n)$ for all $n \ge n_0$ Equivalently: $g(n)$ is $O(f(n))$ g is an asymptotic lower bound on f
$f(n)$ is $\Theta(g(n))$	$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$ g is an asymptotically tight bound on f

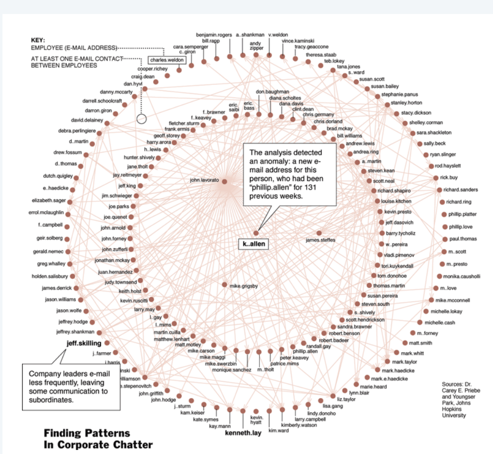
Graphs are everywhere

Some graphs

- Transportation networks: hubs, links, routes
- Communication networks: routing, how many hops, latency/throughput?
- Information networks: WWW, what are important/authoritative pages?
- Social networks: study interaction dynamics, find influencers?

How do we build algorithms to answer these questions?

One week of Enron emails



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Framingham heart study

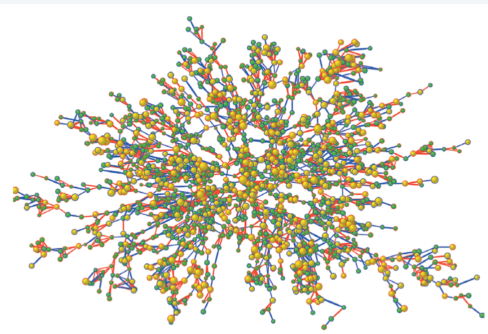

Figure 1. Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000. Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index, ≥30) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

"The Spread of Obesity in a Large Social Network over 32 Years" by Christakis and Fowler in New England Journal of Medicine, 2007

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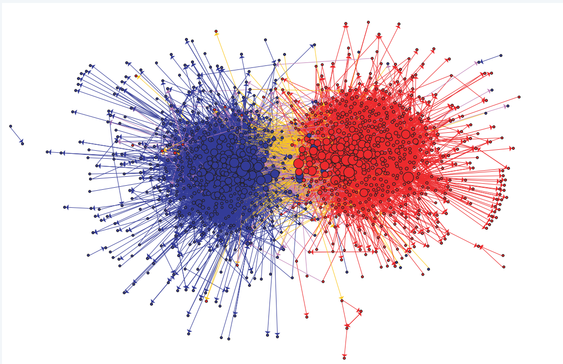
Political blogosphere graph

Node = political blog; edge = link.



The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005

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More applications

- Network science
  - random graphs: various evolution models
  - scale-free, small world
- Analyzing graph evolution in time
  - fake news
  - botnets
- Analyzing programs
  - control flow graph, function call graph
  - state space search (also in games): compute reachable states (configurations) is an error state reachable?

Graphs

A graph is a mathematical representation of a network

- Set of nodes (vertices) V
- Set of pairs of nodes (edges) E (a *relation*)

Graph $G = (V, E)$

Definitions: edge, path

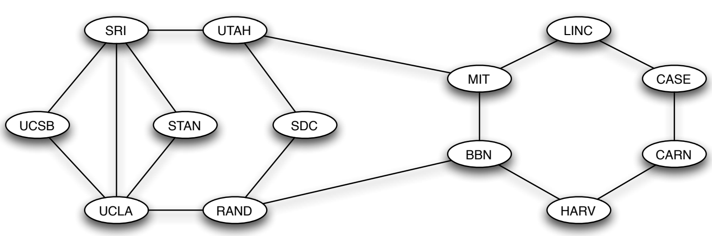
Edge $e = \{u, v\}$ (for an undirected graph)
but usually written $e = (u, v)$

u and v are *neighbors*, *endpoints* of e

A **path** is a sequence $P = v_1, v_2, \ldots, v_{k-1}, v_k$ such that each consecutive pair v_i, v_{i+1} is joined by an edge in G

Called: path "from v_1 to v_k". Or: a v_1–v_k path

Clicker Question 1



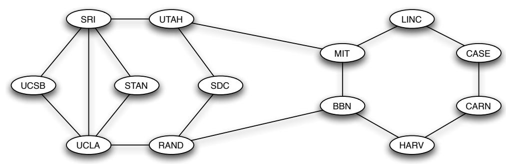
Q: Which is not a path?

1. UCSB - SRI - UTAH
2. LINC - MIT - LINC - CASE
3. UCSB - SRI - STAN - UCLA - UCSB
4. None of the above

Simple path, distance, cycle

- **Simple path**: path where all vertices are distinct
  - Exercise. Prove: If there is a path from u to v then there is a simple path from u to v.
- **Distance** from u to v: minimum number of edges in a u–v path
- **Cycle**: path $v_1, \ldots, v_{k-1}, v_k$ where $v_1 = v_k$ $(k > 1)$
  - **Simple** cycle: no repeated nodes (except first = last)

Connected components

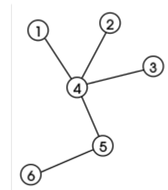


Connected graph = graph with paths between every pair of vertices.

- **Connected component**: subset of nodes U such that
  - A: Paths exist between every pair of nodes in U
  - B: A is not true if any nodes added to U (U is maximal)

Trees

Tree = a connected graph with no cycles



- Q: Is this equivalent to trees you saw in Data Structures?
- A: More or less.
- **Rooted tree**: tree with parent-child relationship
  - Pick root r and "orient" all edges away from root
  - Parent of v = predecessor on path from r to v

Trees

Let G be an undirected graph with n nodes.
Then any two of the following statements implies the third

- G is connected
- G does not contain a cycle
- G has $n - 1$ edges

Directed Graphs

- **Directed graph** $G = (V, E)$
  - *Directed* edge $e = (u, v)$ is now an *ordered* pair
  - e *leaves* u (source) and *enters* v (sink)
- **Directed path, cycle**: same as before, but with directed edges
- **Strongly connected**: directed graph with directed path between every pair of vertices
- Note: graphs undirected if not otherwise specified

Graph Traversal

Thought experiment. World social graph.

- Is it connected?
- If not, how big is largest connected component?
- Is there a path between you and <*some famous person*>?

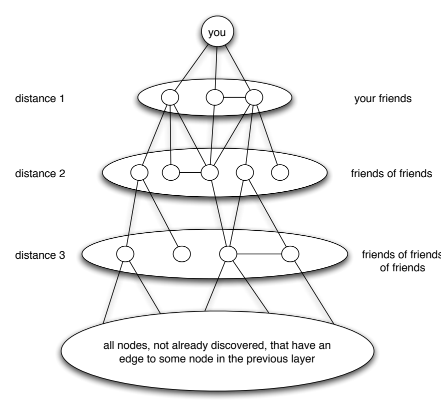
"Six degrees of separation" (everyone connected in at most 6 links?)
Erdös number: coauthorship of scientific papers

How can you tell algorithmically?

Answer: graph traversal! (BFS/DFS)

Breadth-First Search

Explore outward from starting node by distance. "Expanding wave"



Breadth-First Search: Layers

Explore outward from starting node s.

Define layer L_i = all nodes at distance exactly i from s.

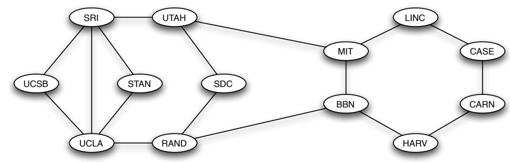
Layers

- $L_0 = \{s\}$
- L_1 = nodes with edge to L_0
- L_2 = nodes with an edge to L_1 that don't belong to L_0 or L_1
- ...
- L_{i+1} = nodes with an edge to L_i that don't belong to any earlier layer.

Observation:
There is a path from s to t if and only if t appears in some layer.

Clicker Question 2

How many nodes are in layer 2, starting a BFS from MIT ?



A) 3

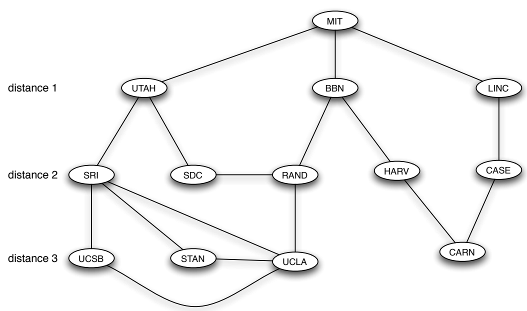
B) 4

C) 5

D) None of the above

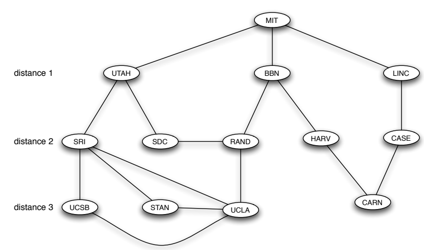
BFS Tree

Exercise: draw the BFS layers for a BFS starting from MIT



We can use BFS to make a tree.

BFS Tree



Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G.
Then the layers of x and y in T differ by at most 1.

Proof?

BFS and non-tree edges

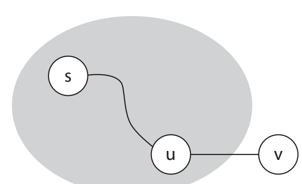
Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G.
Then the layers of x and y in T differ by at most 1.

Proof

- Let (x, y) be an edge
- Suppose $x \in L_i$, $y \in L_j$, and $j > i + 1$
- When BFS visits x, either y is already discovered or not.
  - If y is already discovered, then $j \le i + 1$. Contradiction.
  - Otherwise since $(x, y) \in E$, y is added to L_{i+1}. Contradiction.

A More General Exploration Strategy

To explore the connected component containing s:

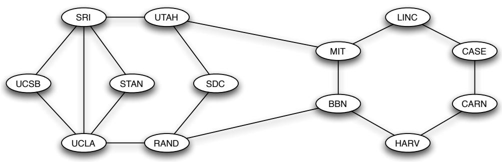


Add any node v for which

- (u, v) is an edge
- u is explored, but v is not

Depth-First Search

Depth-first search (DFS): keep exploring from the most recently added node until you have to backtrack.
Example.



Recursive DFS

DFS(u)
  Mark u as "explored"
  for each edge (u, v) incident to u **do**
    if v is not marked "explored" **then**
      Recursively invoke DFS(v)
    end if
  end for

Exercise: do an example

DFS Tree

Can also extract tree T from DFS.

- $(u, v) \in T$ if v explored from u—i.e., DFS(u) calls DFS(v)

Claim: let T be a depth-first search tree for graph $G = (V, E)$, and let (x, y) be an edge that is in G but not T (a "non-tree edge"). Then either x is an ancestor of y or y is an ancestor of x in T.

Proof?

DFS and Non-tree edges

Claim: let T be a depth-first search tree for graph $G = (V, E)$, and let (x, y) be an edge that is in G but not T (a "non-tree edge"). Then either x is an ancestor of y or y is an ancestor of x in T.

Proof

- Suppose not and suppose that x is reached first by DFS.
- Before leaving x, we must examine (x, y).
- Since $(x, y) \notin T$, y must have been explored by this time.
- But y was not explored when we arrived at x by assumption.
- Thus y was explored during the execution of $\texttt{DFS}(x)$.
- Implies x is ancestor of y.

Exploring *all* Connected Components

How to explore entire graph even if it is disconnected?

while there is some unexplored node s **do**
  BFS(s) ▷ Run BFS starting from s.
  Extract connected component containing s
end while

Usually OK to assume graph is connected.
State if you are doing so and why it does not trivialize the problem.

Running time? What's the running time of BFS?

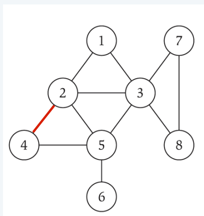
Implementation

- How do we *implement* graph traversal?
  What is the running time?
- Preliminaries
  - Let $m = |E|$ be the number of edges
  - Let $n = |V|$ be the number of nodes
  - Data structure to represent graph? ...

Graph representation: adjacency matrix

n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge

Space proportional to n^2



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Clicker Question 3

An adjacency matrix representation for graph (V, E) with $|V| = n$ takes time

A) $\Theta(n)$ to check if (u, v) is an edge, $\Theta(|E|)$ to traverse all edges

B) $\Theta(n)$ to check if (u, v) is an edge, $\Theta(n^2)$ to traverse all edges

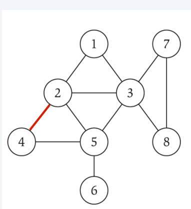
C) $\Theta(1)$ to check if (u, v) is an edge, $\Theta(n^2)$ to traverse all edges

D) $\Theta(1)$ to check if (u, v) is an edge, $\Theta(|E|)$ to traverse all edges

Graph representation: adjacency matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.

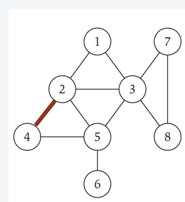


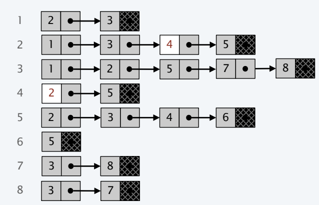
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

slide credit: Kevin Wayne / Pearson

Graph representation: adjacency lists

Adjacency lists. Each node keeps list of neighbors





- Each edge stored twice
- Space? $\Theta(m + n)$
- Checking if (u, v) is an edge?
  $O(\text{degree}(u))$ time (degree = number of neighbors)

Traversal Implementations

Generic approach: maintain set of explored nodes and discovered nodes

- Explored = have seen this node and explored its outgoing edges
- Discovered = the "frontier". Have seen the node, but not explored its outgoing edges.

Generic Graph Traversal

Let A = data structure of discovered nodes

```
Traverse(s)
  Put s in A
  while  A is not empty  do
     Take a node v from A
     if v is not marked "explored" then
        Mark v as "explored"
        for  each edge (v, w) incident to v  do
           Put w in A                         ▷ w is discovered
        end for
     end if
  end while
```

Note: one part of this algorithm seems wasteful. Why?
Can put multiple copies of a single node in A.

BFS: A is a queue (FIFO)   DFS: A is a stack (LIFO)