COMPSCI 311: Introduction to Algorithms

Lecture 21: Reductions and NP-Complete Problems

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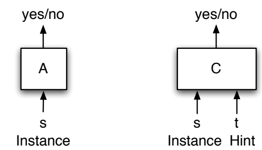
slides credit: Dan Sheldon (adapted)

Review

- P – class of problems with polytime algorithm.
- NP – class of problems with polytime certifier.

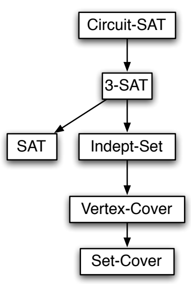


Example

Problem (X)	INDEPENDENT-SET
Instance (s)	Graph G and number k
Algorithm (A)	Try all subsets and check (but not poly-time)
Hint (t)	Which nodes are in the answer?
Certifier (C)	Are those nodes independent and size k?

NP-Complete Problems So Far

Theorem: INDEPENDENTSET, VERTEXCOVER, SETCOVER, SAT, 3-SAT are all NP-Complete.



Clicker Question 1

Which of the following statements is NOT true?

A: SAT $\leq_P$ 3-SAT

B: 3-SAT $\leq_P$ SAT

C: K-SAT $\leq_P$ SAT for all $k \geq 2$

D: K-SAT is NP-complete for all $k \geq 2$

Finding NP-Complete Problems

Want to prove problem X is NP-complete

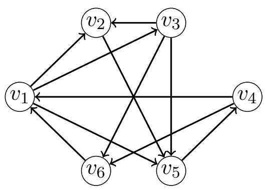
- Check $X \in$ NP.
- Choose known NP-complete problem Y.
- Prove $Y \leq_P X$.
- Usually suffices to do single transformation $s_Y \rightarrow s_X$ s.t.
  - s_X is YES instance iff s_Y is YES instance

Traveling Salesman Problem

- TSP. Given n cities and distance function $d(i, j)$, is there a tour that visits all cities with total distance less than D?
  - Tour: ordering of cities $i_1, i_2, \ldots, i_n$ with $i_1 = 1$
  - Distance is $\sum_{j=1}^{n-1} d(i_j, i_{j+1}) + d(i_n, 1)$
- Applications: traveling salesperson, moving robotic arms
- Let's prove a simpler problem is NP-complete, and then use it to show TSP is NP-complete.

Hamiltonian Cycle Problem

- HamCycle – Hamiltonian Cycle. Given directed graph $G = (V, E)$, is there a cycle that visits each vertex exactly once?



- $v_1, v_3, v_2, v_5, v_4, v_6$ is a Hamiltonian Cycle

Ham-Cycle

Theorem. Ham-Cycle is NP-Complete.

- It is in NP.
- Need to reduce from some NP-Complete problem. Which one?

Claim. 3-SAT $\leq_P$ Ham-Cycle.

Reduction has two main parts.

- Make a graph with 2^n Hamiltonian cycles, one per assignment.
- Augment graph with clauses to invalidate assignments.

Reduction: Graph skeleton for truth assignments

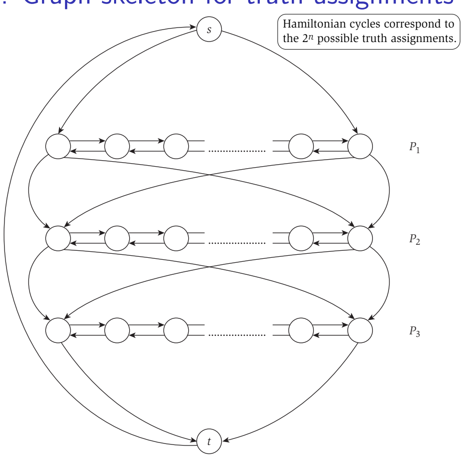


Figure 8.7 The reduction from 3-SAT to Hamiltonian Cycle: part 1.

Reduction: Graph skeleton with clause constraints

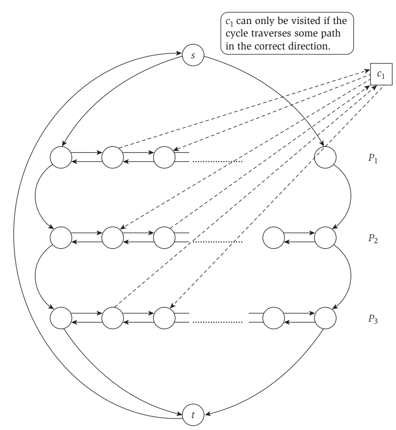


Figure 8.8 The reduction from 3-SAT to Hamiltonian Cycle: part 2.

Reduction: Skeleton Construction

- n rows (bidirected paths) $P_1, \ldots, P_n$ (one per variable)
- Row has $3m + 3$ vertices, connected to neighbors in forward/backward direction
- First and last vertex of row i connected to first and last of $i + 1$.
- Source s connected to first and last of row 1.
- First and last of row n connected to t.
- Edge (t, s)
- Skeleton has 2^n possible Hamiltonian Cycles, corresponding to truth assignments to $x_1, \ldots, x_n$
  - Traverse P_i L to R $\iff x_i = 1$
  - Traverse P_i R to L $\iff x_i = 0$

Reduction: Clause Gadgets

For each clause C_ℓ construct gadget to restrict possible truth assignments

- New node c_ℓ
- If $x_i \in C_\ell$
  - Add edges $(v_{i,3\ell}, c_\ell)$ and $(c_\ell, v_{i,3\ell+1})$
  - c_ℓ can be visited during L to R traversal of P_i
- If $\neg x_i \in C_\ell$
  - Add edges $(v_{i,3\ell+1}, c_\ell)$ and $(c_\ell, v_{i,3\ell})$
  - c_ℓ can be visited during R to L traversal of P_i

Proof of Correctness

Given a satisfying assignment, construct Hamiltonian Cycle

- If $x_i = 1$ traverse P_i from $L \to R$, else $R \to L$.
- Each C_ℓ is satisfied, so one path P_i is traversed in the correct direction to "splice" c_ℓ into our cycle
- The result is a Hamiltonian Cycle

Given Hamiltonian cycle, construct satisfying assignment:

- If cycle visits c_ℓ from row i, it will also leave to row i because of "buffer" nodes
- Therefore, ignoring clause nodes, cycle traverses each row completely from $L \to R$ or $R \to L$
- Set $x_i = 1$ if P_i traversed $L \to R$, else $x_i = 0$
- Every node c_j visited $\Rightarrow$ every clause C_j is satisfied

Traveling Salesman

TSP. Given n cities and distance function $d(i, j)$, is there a tour that visits all cities with total distance less than D?

Theorem. TSP is NP-Complete

- Clearly in NP.
- Reduction? From Ham-Cycle

Reduction from Ham-Cycle to TSP

Given HamCycle instance $G = (V, E)$ make TSP instance

- One city per vertex
- $d(v_i, v_j) = 1$ if $(v_i, v_j) \in E$, else 2

Claim: there is a tour of distance $\leq n$ if and only if G has a Hamiltonian cycle

- A Hamiltonian cycle clearly gives a tour of length n
- A tour of length n must travel n hops of length 1, which corresponds to a Hamiltonian cycle

Ham-Path

Similar to Hamiltonian Cycle, visit every vertex exactly once.

Theorem. Ham-Path is NP-Complete.

Two proofs.

- Modify 3-SAT to Ham-Cycle reduction.
- Reduce from Ham-Cycle directly.

Clicker Question 2

Suppose now I want to reduce Ham-Path to Ham-Cycle. Which of the following statements is true?

A: Trivial: any cycle with all nodes is also a path with all nodes

B: Ham-Path $\not\leq_P$ Ham-Cycle since not all paths are cycles

C: Ham-Path $\not\leq_P$ Ham-Cycle since there are some graphs that have a Hamiltonian path but no Hamiltonian cycle

D: None of the above

NP-Complete Problems