COMPSCI 311: Introduction to Algorithms

Lecture 15: Dynamic Programming – Shortest Paths

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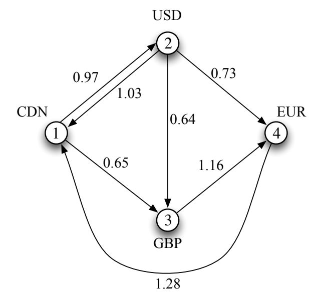
slides credit: Dan Sheldon (adapted)

Currency Trading



- **Problem**: given directed graph with exchange rate r_e on edge e, find best exchange rate $s \to t$, i.e., path P with maximum product $\prod_{e \in P} r_e$ over edges
- **Assumption** (no arbitrage): no cycles C with $\prod_{e \in C} r_e > 1$.

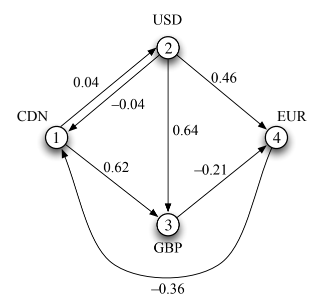
Shortest Paths

- We know how to find minimum sum, not maximum product,but
  - logarithm of product is sum of logs
  - maximize x means minimize -x
- Let $c_e = -\log r_e$ be the *cost* of edge e
- Let the path cost be the negative log of the path exchange rate.

$$\begin{aligned} \text{cost}(P) &= -\log \prod_{e \in P} r_e \\ &= \sum_{e \in P} (-\log r_e) \\ &= \sum_{e \in P} c_e \end{aligned}$$

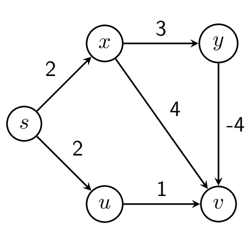
- Equivalent problem: find the $s \to t$ path of minimum cost

Currency Trading



- Negative edge weights!
- **Problem**: given a graph with possibly negative edge weights, find shortest $s \to t$ path
- **Assumption**: no cycle C with $\sum_{e \in C} c_e < 0$. Why?

Dijkstra's Algorithm: Negative Edge Behavior



What is the shortest path value the algorithm finds for $d(s, v)$?

Clicker Question 1

When run on a graph with negative edges, Dijkstra's algorithm:

A: Does not give the right value if shortest path has negative edge.

B: May give the right value even if the shortest path has a negative edge.

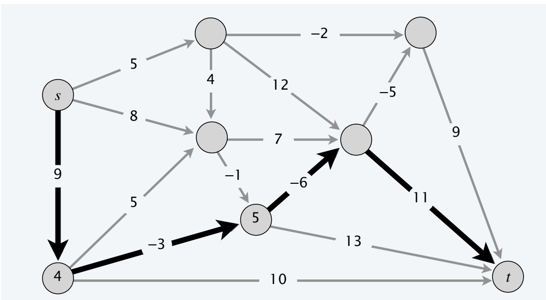
C: Does not give the right value if the target node is first reached through a positive edge.

D: Gives the right value if the target node is first reached through a negative edge.

Choose the most precise answer!

Clicker Question 2

In the following graph, which is the value of the shortest $s \to t$ path found by Dijkstra's algorithm?



A: 26
B: 20
C: 12
D: 11

Bellman-Ford Algorithm: Setup

Consider shortest paths from any node to a given target node t (single-destination shortest paths)

Like single-source, but destination more relevant e.g., in routing

Consider paths with increasing number of edges to target

Fact. If no negative cycles, shortest path has at most $n-1$ edges. Why?

Path with $\geq n$ edges has $\geq n+1$ nodes: would repeat some node, thus cycle!

Clicker Question 3

For shortest paths from any v to a fixed t, we'd like to compute $\mathrm{OPT}(i+1, v)$ from $\mathrm{OPT}(i, v)$, by incrementing the edge count i.

If we find a better path starting with edge (v, w), we want to update

$\mathrm{OPT}(i+1, v) = c_{v,w} + \mathrm{OPT}(i, w)$

Should $\mathrm{OPT}(i, v)$ mean the optimal cost from v to t

A: on a path with i edges

B: on a path with at most i edges

Bellman-Ford Recurrence

- Let $\mathrm{OPT}(i, v)$ be cost of shortest $v \to t$ path with at most i edges.
- **Recursive principle**: let P be the optimal $v \to t$ path using at most $i+1$ edges.
  - If P uses at most i edges, then $\mathrm{OPT}(i+1, v) = \mathrm{OPT}(i, v)$.
  - Else $P = v \to w \to t$ where $w \to t$ path uses at most i edges.

$$\mathrm{OPT}(i+1, v) = c_{v,w} + \mathrm{OPT}(i, w)$$

$$\mathrm{OPT}(i, v) = \min\left\{\mathrm{OPT}(i-1, v),\ \ \min_{w \in V}\{c_{v,w} + \mathrm{OPT}(i-1, w)\}\right\}$$

Bellman-Ford Algorithm

$$\mathrm{OPT}(i, v) = \min\left\{\mathrm{OPT}(i-1, v),\ \ \min_{w \in V}\{c_{v,w} + \mathrm{OPT}(i-1, w)\}\right\}$$

Shortest-Path(G, s, t)
  n = number of nodes in G
  Create array M of size $n \times n$
  Set $M[0, t] = 0$ and $M[0, v] = \infty$ for all other v
  for $i = 1$ to $n-1$ **do**
    for all nodes v in any order **do**
      Compute $M[i, v]$ using the recurrence above
    end for
  end for

Running time? $O(n^3)$. Better analysis: $O(mn)$.

Shortest paths with negative weights: practical improvements

Space optimization. Maintain two 1D arrays (instead of 2D array).

- $d[v]$ = length of a shortest $v \leadsto t$ path that we have found so far.
- $successor[v]$ = next node on a $v \leadsto t$ path.

Performance optimization. If $d[w]$ was not updated in iteration $i-1$, then no reason to consider edges entering w in iteration i.

slide credit: Kevin Wayne / Pearson

Bellman-Ford-Moore: efficient implementation

BELLMAN–FORD–MOORE(V, E, c, t)

FOREACH node $v \in V$:
  $d[v] \leftarrow \infty$.
  $successor[v] \leftarrow null$.
$d[t] \leftarrow 0$.
FOR i = 1 TO n – 1
  FOREACH node $w \in V$:
    IF ($d[w]$ was updated in previous pass)
      FOREACH edge $(v, w) \in E$:
        IF ($d[v] > d[w] + \ell_{vw}$)
          $d[v] \leftarrow d[w] + \ell_{vw}$.
          $successor[v] \leftarrow w$.
  IF (no $d[\cdot]$ value changed in pass i) STOP.

pass i
$O(m)$ time

Clicker Question 4

Consider a directed graph with arbitrary edge weights.

Then, at every step of running Bellman-Ford

A: Following successor[v] pointers gives a $v \to t$ path

B: The length of the successor[v] path is d[v]

C: Both A and B

D: Neither A nor B

A: No, d[v] can be one iteration behind, if successor[v] = w same, but d[w] just got updated.
B: No, for negative-weight cycles (next slide)

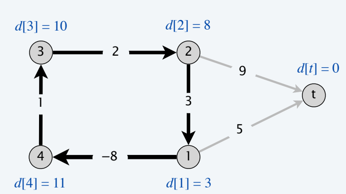
Bellman-Ford-Moore: analysis

Claim. ~~Throughout Bellman–Ford–Moore, following the $successor[v]$ pointers gives a directed path from v to t of length $d[v]$.~~

Counterexample. Claim is false!

- Length of successor $v \leadsto t$ path may be strictly shorter than $d[v]$.
- If negative cycle, successor graph may have directed cycles.

consider nodes in order: t, 1, 2, 3, 4



Detecting Negative-Weight Cycles

We've seen that absent negative-weight cycles, a shortest path has at most $n - 1$ edges.

Run for one extra iteration (n). If OPT(n, v) decreases for some v, we have a negative-weight cycle! (why?)

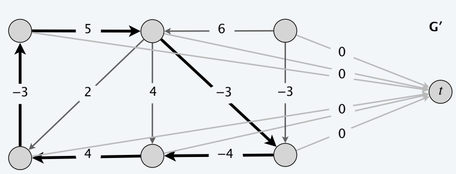
But this is only over paths to a fixed target node t. How to cover the entire graph?

Add dummy sink node with zero-cost edges from all nodes.
Use this as target (all nodes are predecessors, will be covered).

Detecting negative cycles

Theorem 4. Can find a negative cycle in $\Theta(mn)$ time and $\Theta(n^2)$ space.
Pf.

- Add new sink node t and connect all nodes to t with 0-length edge.
- G has a negative cycle iff G' has a negative cycle.
- Case 1. [$OPT(n, v) = OPT(n - 1, v)$ for every node v]
  By Lemma 7, no negative cycles.
- Case 2. [$OPT(n, v) < OPT(n - 1, v)$ for some node v]
  Using proof of Lemma 8, can extract negative cycle from $v \leadsto t$ path.
  (cycle cannot contain t since no edge leaves t) ▪



Detecting negative cycles

Theorem 5. Can find a negative cycle in $O(mn)$ time and $O(n)$ extra space.
Pf.

- Run Bellman–Ford–Moore on G' for $n' = n + 1$ passes (instead of $n' - 1$).
- If no $d[v]$ values updated in pass n', then no negative cycles.
- Otherwise, suppose $d[s]$ updated in pass n'.
- Define $pass(v)$ = last pass in which $d[v]$ was updated.
- Observe $pass(s) = n'$ and $pass(successor[v]) \geq pass(v) - 1$ for each v.
- Following successor pointers, we must eventually repeat a node.
- Lemma 6 $\Rightarrow$ the corresponding cycle is a negative cycle. ▪

Remark. See p. 304 for improved version and early termination rule.
(Tarjan's subtree disassembly trick)