

## Homework 2

Released 9/20/2018

Due 10/2/2018 11:59pm in Gradescope

**Instructions.** You may work in groups, but you must individually write your solutions yourself. List your collaborators on your submission.

If you are asked to design an algorithm as part of a homework problem, please provide: (a) the pseudocode for the algorithm, (b) an explanation of the intuition for the algorithm, (c) a proof of correctness, (d) the running time of your algorithm and (e) justification for your running time analysis.

**Submissions.** Please submit a PDF file. You may submit a scanned handwritten document, but a typed submission is preferred. It will be very helpful if in your submission each question starts on a new page.

1. **(10 points) Spanning Trees and Shortest Paths.** Consider two undirected graphs  $G$  and  $G'$  with the same structure of nodes and edges. This first graph has non-negative edge lengths  $\ell(e) \geq 0$ . In the second graph, each edge length is increased by one, so  $\ell'(e) = \ell(e) + 1$ .

(a) If  $T$  is a minimum spanning tree in the first graph, is it always a minimum spanning tree in the second graph? If it is, then prove it, otherwise give a counterexample.

(b) Suppose we have computed the shortest paths from some fixed node  $s \in V$  and let  $t$  be any other node. If  $p$  is a shortest  $s \rightarrow t$  path in the first graph, is it always a shortest  $s \rightarrow t$  path in the second graph? If it is, then prove it, otherwise give a counterexample.

2. **(15 points) Spanning and DFS tree** (Manber Ex 7.3). Consider a connected undirected graph  $G = (V, E)$ , a spanning tree  $T$  of  $G$  and a vertex  $v$ . Design an  $O(|V| + |E|)$  algorithm to determine whether  $T$  is a valid DFS tree of  $G$  rooted at  $v$ , that is, determine whether  $T$  can be the output of DFS starting from  $v$ , under some order of the edges.

3. **(20 points) Visiting all edges.** Suppose you have a connected network of two-way streets.

(a) Show that you can drive along these streets so that you visit all streets and you drive along each side of every street exactly once.

(b) Suppose you drive in such a way that at each intersection, you do not leave by the street you used to enter that intersection unless you have previously left via all other streets from that intersection. Does this give a valid way to solve part (a)? If yes, prove this, and if not, construct a counterexample.

4. **(20 points) Counting change**

Consider the problem of making change for  $n$  cents using the least number of coins. Assume that each coin's value is an integer and the lowest denomination coin is  $c_0 = 1$  (a penny).

a) Give a (simple but general) sufficient condition for the coin denominations  $c_0 < c_1 < \dots < c_k$  so that the obvious greedy algorithm of making change gives an optimal solution, and prove this.

b) Show that the greedy algorithm also works for the the U.S. coin system (quarters, dimes, nickels, pennies). If this does not fall under your condition at point (a), argue why it still works.

5. **(15 points) Scheduling to minimize average completion time** (CLRS, problem 16-2)

You are given a set  $S = \{a_1, a_2, \dots, a_n\}$  of tasks, where task  $a_i$  requires  $p_i$  units of processing time to complete, and cannot be interrupted once started. The computer can run only one task at a time. Let  $c_i$  be the completion time of task  $a_i$ , that is, the time at which task  $a_i$  completes processing. Give an algorithm that minimizes the average completion time, that is,  $\frac{1}{n} \sum_{i=1}^n c_i$ .

**6. (20 points) Spanning Tree (K&T Ch4 Ex9)**

Design a spanning tree for which the *most expensive* edge is as cheap as possible. Let  $G = (V, E)$  be a connected graph with  $n$  vertices,  $m$  edges, and positive edge costs assumed all distinct. Let  $T = (V, E')$  be a spanning tree of  $G$ ; we define the *bottleneck edge* of  $T$  to be the edge of  $T$  with the greatest cost.

A spanning tree  $T$  of  $G$  is a minimum-bottleneck spanning tree if there is no spanning tree  $T'$  of  $G$  with a cheaper bottleneck edge.

- a) Is every minimum-bottleneck tree a minimum spanning tree of  $G$ ? Prove or give a counterexample.
- b) Is every minimum spanning tree a minimum-bottleneck tree of  $G$ ? Prove or give a counterexample.

**7. (0 points).** How long did it take you to complete this assignment?