CS250 Fall 2020 Midterm 2 Solutions, Evening Exam

2. *Use the Principle of Mathematical Induction to prove that for every natural n, $2^n 3^n$ divides $(3n)!$.*

Base case: $P(0)$ is true, since $2^0 \cdot 3^0 = 1$ divides $(3 \cdot 0)! = 1$.
Inductive step. Assume $P(n)$ true for some $n \geq 0$, that is, $(3n)! = k \cdot 2^n \cdot 3^n$ for some $k \in \mathbb{N}$. Then $(3(n+1))! = (3n)! \cdot (3n+1)(3n+2)(3n+3) = k \cdot 2^n \cdot 3^n \cdot (3n+1)(3n+2) \cdot 3(n+1)$, and we have the needed extra factor of 3. Also, $3n+1$ and $3n+2$ are consecutive numbers, so one of them is even and gives us another factor of 2, which shows $P(n+1)$.

3. *Let $P(n)$ be the statement that Amy can pay a present of n cents by using just 3-cent and 7-cent coins. a) Prove $P(n)$ for all $n \geq 12$ by mathematical induction.*

$P(12)$ is true since $12 = 3 \cdot 4$: four 3-cent coins and zero 7-cent coins make a 12-cent present.

Now we show that $P(k) \to P(k+1)$. Suppose $P(k)$ is true, i.e., a k-cent present can be formed. There are two cases to consider: (1) at least two 3-cent coins were used in k-cent present, or (2) at most one 3-cent coin was used in k-cent present. We consider each of these in turn.

Case 1 If two 3-cent coins were used in a k-cent present, replace two 3-cent coins with one 7-cent coin to form a $k - 3 \cdot 2 + 7 \cdot 1 = k + 1$ cent present.

Case 2 If at most one 3-cent coin was used, since $k > 3 + 7$, at least two 7-cent coins were used. Replace two 7-cent coins with five 3-cent coins to form $k - 7 \cdot 2 + 3 \cdot 5 = k + 1$ cents.

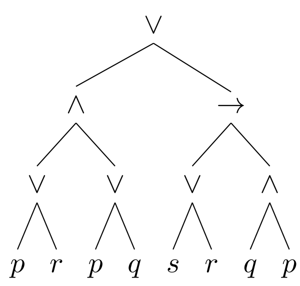
b) Prove $P(n)$ for all $n \geq 12$ by strong induction.

We prove $P(12)$, $P(13)$ and $P(14)$ for the base case.

$P(12)$ holds because $12 = 3 \cdot 4 + 7 \cdot 0$. $P(13)$ holds because $13 = 3 \cdot 2 + 7 \cdot 1$; and $P(14)$ holds because $14 = 3 \cdot 0 + 7 \cdot 2$. We state and prove the inductive step to complete the strong induction proof.

Let $k \geq 14$. Assume $P(j)$ for all j such that $12 \leq j \leq k$. We prove that $P(k+1)$. Since $k \geq 14$, it follows that $k - 2 \geq 12$, and we know that $P(k-2)$ is true. So a $k-2$ cent present can be formed using 3-cent and 7-cent coins. Add a 3-cent coin to form a $(k-2) + 3 = k + 1$ cent present. Therefore, $P(k+1)$.

4. *Draw the expression tree (parse tree) from the postfix string "$p\ r \lor p\ q \lor \land\ s\ r \lor q\ p \land \to \lor$".*



We identify four groups of the form *proposition proposition operator*, which form expression subtrees. We continue finding groups of the form *expression expression operator*, building larger subtrees.

5. *Given the recurrence $a_0 = 0$, $a_1 = 1$, $a_{n+1} = 3a_n - 2a_{n-1}$ for $n > 0$, find and prove a statement about when a_n is divisible by 7.*

We unroll $a_n = 3a_{n-1} - 2a_{n-2} = 3(3a_{n-2} - 2a_{n-3}) - 2a_{n-2} = 7a_{n-2} - 6a_{n-3}$, which holds for $n \geq 3$. We get $a_n \equiv -6a_{n-3} \equiv (7-6)a_{n-3} \equiv a_{n-3} \pmod 7$. We claim $a_n \equiv 0 \pmod 7 \leftrightarrow n \equiv 0 \pmod 3$.

This is true for the base cases $n = 0, 1, 2$: $a_0 = 0$, $a_1 = 1$, $a_2 = 3$. We prove the statement by strong induction for $n \geq 3$, assuming it true for all $n' < n$.
We have $a_n \equiv 0 \pmod 7 \leftrightarrow a_{n-3} \equiv 0 \pmod 7 \leftrightarrow n - 3 \equiv 0 \pmod 3 \leftrightarrow n \equiv 0 \pmod 3$, with the middle equivalence by inductive hypothesis.

6. *Fibonacci words are strings of 0 and 1 defined as follows: $S_0 = 0$, $S_1 = 01$, $S_{n+1} = S_n \cdot S_{n-1}$, where $\cdot$ means concatenation.*

Find a recurrence for the number Z_n of occurrences of the string 10 *in S_n. First, find an even/odd recurrence and argue why it holds. Then find and prove a closed formula for Z_n in terms of the Fibonacci numbers.*

The number of 10 strings in $S_n S_{n-1}$ is the sum of the number of 10 strings in each of Sn and S_{n-1}, with possibly one more string formed when joining. All strings start with 0, by induction, since the first digit of S_{n+1} is the first digit of S_n. By construction, the last digit of S_{n+1} is the last digit of S_{n-1}, for $n > 0$. By odd-even induction, with the base cases of $S_0 = 0$ and $S_1 = 01$, the last digit of S_n is $n \bmod 2$. Thus, a 10 string is formed in S_{n+1} exactly when n is odd.

We get the recurrences $Z_{n+1} = Z_n + Z_{n-1}$ for $n > 0$ even, and $Z_{n+1} = Z_n + Z_{n-1} + 1$ for n odd, with $Z_0 = Z_1 = 0$. Substituting each recurrence in the other, we get $Z_{n+1} = 2Z_{n-1} + Z_{n-2} + 1$ for $n > 1$ (the problem asked for a recurrence by getting an odd/even recurrence fist, but very few wrote this; it was not penalized). We could also write $Z_{n+1} = Z_n + Z_{n-1} + n \bmod 2$, for $n > 0$.

We observe that $Z_n = F_n - n \bmod 2$, or alternatively, $Z_n = F_n + ((-1)^n - 1)/2$. This can also be obtained by solving the above characteristic equation $x^3 - 2x - 1 = (x+1)(x^2 - x - 1) = 0$, which has the golden ratio (Fibonacci) roots, and -1, which gives us the alternating term.

We prove this by induction, verifying the base case, $Z_0 = F_0$, $Z_1 = F_1 - 1$, and adding: $Z_{n+1} = (F_n - n \bmod 2) + (F_{n-1} - (n-1) \bmod 2) + n \bmod 2 = F_{n+1} - (n-1) \bmod 2 = F_{n+1} - (n+1) \bmod 2$.

CS250 Fall 2020 Midterm 2 Solutions, Morning Exam

2. *Use the Principle of Mathematical Induction to prove that $n^{2n+1} \geq n!^2$ for every integer $n \geq 1$.*

The base case holds, $1^3 \geq 1!^2$, both are 1. Assume the inequality holds for $n \geq 1$ and prove it for $n+1$. We have: $(n+1)^{2n+3} = (n+1)^{2n+1}(n+1)^2 > n^{2n+1}(n+1)^2 \geq n!^2(n+1)^2 = (n+1)!$, with the last inequality from the induction hypothesis. This completes the inductive step.

3. *Let $P(n)$ be the statement that a postage of n cents can be formed using just* 5*-cent and* 8*-cent stamps.*
a) Prove $P(n)$ for all $n \geq 28$ by mathematical induction.

$P(28)$ is true: $28 = 8 \cdot 1 + 5 \cdot 4$, so one eight-cent stamp and four five-cent stamps make 28 cents postage.

Now we show that $P(k) \to P(k+1)$. Suppose $P(k)$ is true, i.e., k-cent postage can be formed. There are two cases to consider: (1) at least three 8-cent stamps were used in k-cent postage, or (2) at most two 8-cent stamps were used in k-cent postage. We consider each of these in turn.

Case 1 If three 8-cent stamps were used in k-cent postage, replace three 8-cent stamps with five 5-cent stamps to form $k - 8 \cdot 3 + 5 \cdot 5 = k + 1$ cents postage.

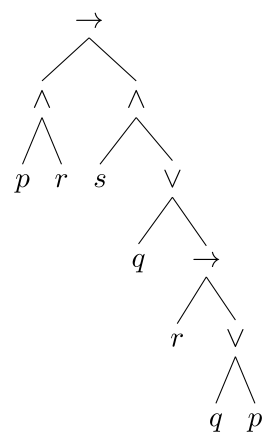
Case 2 If at most two 8-cent stamps were used, since $k > 26 = 2 \cdot 8 + 2 \cdot 5$, at least three 5-cent stamps were used. Replace three 5-cent stamps with two 8-cent stamps to form $k - 5 \cdot 3 + 8 \cdot 2 = k + 1$ cents postage.

b) Prove $P(n)$ for all $n \geq 28$ by strong induction.

We prove $P(28)$, $P(29)$, $P(30)$, $P(31)$ and $P(32)$ for the base case. . We have $28 = 8 \cdot 1 + 5 \cdot 4$, $29 = 8 \cdot 3 + 5 \cdot 1$, $30 = 5 \cdot 6$, $31 = 8 \cdot 2 + 5 \cdot 3$, and $32 = 8 \cdot 4$, so all statements hold. Now, we state and prove the inductive step to complete the strong induction proof.

Let $k \geq 32$. Assume $P(j)$ for all j such that $28 \leq j \leq k$. We prove $P(k+1)$. Since $k \geq 32$, we have $k-4 \geq 28$, and we know $P(k-4)$ is true. So $k-4$ cent postage can be formed using 5-cent and 8-cent stamps. Add a 5-cent stamp to form $(k-4)+5 = k+1$ cent postage. Therefore, $P(k+1)$.

4. *Draw the expression tree (parse tree) from the prefix string "* $\rightarrow \land\ p\ r \land s \lor q \rightarrow r \lor q\ p$*".*



The easiest solution is to process the string recursively, and for each operator, construct the left and right subtree.
A more laborious option is to repeatedly process the string bottom up and group triples of the form *operator expression expression* to an expression, starting with $\land\ p\ q$ and $\lor\ q\ p$.

5. *Given the recurrence $a_0 = 0$, $a_1 = 1$, $a_{n+1} = 2a_n + 3a_{n-1}$ for $n > 0$, find and prove a statement about when a_n is divisible by 7.*

We unroll $a_n = 2a_{n-1} + 3a_{n-2} = 2(2a_{n-2} + 3a_{n-3}) + 3a_{n-2} = 7a_{n-2} + 6a_{n-3}$. Indices must be ≥ 0, so $n \geq 3$. We get $a_n \equiv 6a_{n-3} \pmod 7$. This gives us $a_n \equiv 0 \pmod 3 \leftrightarrow a_{n-3} \equiv 0 \pmod 3$, since $\gcd(6,7) = 1$ and 6 has an inverse mod 7. We claim $a_n \equiv 0 \pmod 7 \leftrightarrow n \equiv 0 \pmod 3$.

This is true for the base cases $n = 0, 1, 2$: $a_0 = 0$, $a_1 = 1$, $a_2 = 2$. We prove the statement by strong induction for $n \geq 3$, assuming it true for all $n' < n$.
We have $a_n \equiv 0 \pmod 7 \leftrightarrow a_{n-3} \equiv 0 \pmod 7 \leftrightarrow n-3 \equiv 0 \pmod 3 \leftrightarrow n \equiv 0 \pmod 3$, with the middle equivalence by inductive hypothesis.

6. *Fibonacci words are strings of 0 and 1 defined as follows: $S_0 = 0$, $S_1 = 01$, $S_{n+1} = S_n \cdot S_{n-1}$, where $\cdot$ means concatenation.*

Find a recurrence for the number C_n of occurrences of the string 01 in S_n. First, find an even/odd recurrence and argue why it holds. Then, find a single recurrence valid for all n. Finally, find and prove a closed formula for C_n in terms of the Fibonacci numbers.

We see immediately by induction that S_n starts with 0 for any n, since this is true for S_0 and S_1, and S_{n+1} starts with the same digit as S_n. Therefore, the number of 01 strings in S_{n+1} is the sum of the number of 10 strings in each of S_n and S_{n-1}. No extra string is created by joining, since S_{n-1} starts with 0. We get the recurrence $C_{n+1} = C_n + C_{n-1}$, for $n > 0$, with $C_0 = 0$, $C_1 = 1$. There is no need for separate even-odd recurrences. Since C_n has the same recurrence and initial values as the Fibonacci sequence, we have $C_n = F_n$ for all n.