COMPSCI 250 Introduction to Computation
Solutions to First Midterm, Spring 2019

Question 1 (35):

a. (10p) Translate the following statements, using predicates with arguments over a set D of dogs. Clearly define what your predicates mean.

We use the following predicates (all arguments are of type D (dogs):

$H(x)$	dog x is happy
$O(x,y)$	dog x is the offspring of dog y
$P(x)$	dog x can play in the snow
$T(x)$	dog x is thick-furred

(3p) (A1) Every dog is happy if all its offspring can play in the snow.
$\forall x : (\forall y : (O(y,x) \to P(y)) \to H(x))$
It is wrong to translate placing all quantifiers in front: $\cancel{\forall x : \forall y : ((O(y,x) \wedge P(y)) \to H(x))}$.
This really says that *one* playful offspring y is enough to make x happy (different from A1).
When translating, keep the sentence structure: $\forall x : (\textit{premise for } x \to H(x))$. The premise is: all offspring of x can play in the snow, $\forall y : (O(y,x) \to P(y))$. Transforming the implication gives $\exists y$ for the correct version, while the wrong one has $\forall y$:
(A1): $\forall x : (\neg(\forall y : (O(y,x) \to P(y))) \vee H(x)) = \forall x : \exists y : ((O(y,x) \wedge \neg P(y)) \vee H(x))$.

(1p) (A2) Thick-furred dogs can play in the snow.
$\forall x : (T(x) \to P(x))$

(3p) (A3) A dog is thick-furred if it is the offspring of at least one thick-furred dog.
$\forall y : (\exists x : (O(y,x) \wedge T(x)) \to T(y))$ or equivalently $\forall x : \forall y : ((T(x) \wedge O(y,x)) \to T(y))$.

(2p) (A4) Every thick-furred dog is the offspring of some thick-furred dog.
$\forall y : (T(y) \to \exists x : (T(x) \wedge O(y,x)))$

(1p) (C) All thick-furred dogs are happy.
$\forall x : (T(x) \to H(x))$

b. (15p) Assuming statements A1–A3 are all true, prove that statement (C) is true.
Make it clear every time you use a quantifier proof rule.

(5) $T(t)$	Premise, t arbitrary
(6) $O(z,t)$	Premise, z arbitrary
(7) $O(z,t) \wedge T(t)$	Conjunction
(8) $\exists x : (O(z,x) \wedge T(x))$	Existence (7)
(9) $(\exists x : (O(z,x) \wedge T(x))) \to T(z)$	Specification (A3, $y = z$)
(10) $T(z)$	Modus Ponens (8, 9)
(11) $T(z) \to P(z)$	Specification (A2, $x = z$)
(12) $P(z)$	Modus Ponens (10, 11)
(13) $O(z,t) \to P(z)$	Deduction (6 ,13)
(14) $\forall y : (O(y,t) \to P(y))$	Generalization (13)
(15) $(\forall y : ((O(y,t) \to P(y))) \to H(t)$	Specification (A1, $x = t$)
(16) $H(t)$	Modus Ponens (14, 15)
(17) $T(t) \to H(t)$	Deduction (5, 16)
(18) $\forall x : (T(x) \to H(x))$	Generalization (17)

In words: we start with an arbitrary thick-furred dog t and one of its offspring z. (A3) tells us that z is thick-furred and therefore by (A2) can play in the snow. Since z was arbitrary, this is true for all offspring of t who can thus play in the snow, which by (A1) makes t happy. Generalizing for arbitrary t gives us that all thick-furred dogs are happy.

c. (6p) Does A3 imply A4? Does A4 imply A3? Prove each statement or give a counterexample (a set of dogs with corresponding values for the *offspring* and *thick-furred* predicates).

The wording and quantifier structure tell us (A3) and (A4) mean different things.
Consider a set D with just one dog d, who is thick-furred and is not anyone's offspring. A3 is trivially true, since d is the only dog, and $T(d)$ holds. A4 is false, since d is noone's offspring. Thus, A3 does not imply A4.
We now make A4 true and A3 false. Since every thick-furred dog has a parent, we take an infinite set of dogs indexed by naturals: $D = \{d_n \mid n \geq 0\}$. Make all dogs thick-furred except 0: $T(d_n) = (n > 0)$, and let each dog d_n be the offspring of d_{n+1}: $O(d_n, d_{n+1})$ for $n \geq 0$. Then A4 is true, but A3 is false (due to dog 0), so A4 does not imply A3.

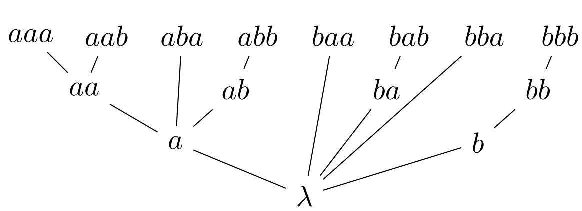
d. (4p) Do A1, A2 and A4 together imply C? Prove or give a counterexample.
The previous point suggests the implication may not hold. Consider the second universe of dogs from point (c), which satisfies A3. Make every dog except 0 able to play in the snow, and every dog except 1 happy. A2 holds for all dogs, since d_0 is not thick-furred, and likewise A1, since d_1's offspring (d_0) cannot play in the snow. C does not hold because d_1 is hot happy. (Note that since d_0 has no offspring, it vacuously satisfies A1 whether it's happy or not).

Question 2 (20):

Consider the set of strings over the alphabet of lowercase letters. Define the relation $R(u, v)$ if $v = uw$, and w contains no letters that come before any of the letters of u in the alphabet.
For example, $R(ba, bab)$ holds, but not $R(ac, acbc)$.

a. (10XC) Prove that R is a partial order on strings. You need not do quantifier proofs, but use notation that makes your reasoning as clear and rigorous as possible.
$R(v, v)$ clearly holds, since $v = v\lambda$ and λ contains no letters at all.
If $R(u, v) \wedge R(v, u)$, then $v = uw$ and $u = vx$, so $v = vxw$, therefore $x = w = \lambda$, and $u = v$.
Assume $R(u, v)$, with $v = ux$, and $R(v, w)$ with $w = vy$. Then $w = uxy$. Moreover, denoting alphabetical ordering by $\leq$, we have: for every letter $\ell_1 \in u, \ell_2 \in x, \ell_3 \in y$, $R(u, v)$ implies $\ell_2 \geq \ell_1$, and $R(v, w)$ implies $\ell_3 \geq \ell_1$, so we have all conditions for $R(u, w)$.

b. (10p) Now consider the same relation, but over the set of strings of at most three letters with the alphabet $\{a, b\}$. Draw the Hasse diagram for the relation R over this set.
The letter ordering constraint means that some strings are only preceded in the order by λ, e.g. ba, baa, bba, thus, we also have lines going across several levels in the diagram.



Question 3 (30):

The following are fifteen true/false or multiple choice questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

a. The proposition $(p \leftrightarrow q) \vee (r \wedge s)$ is true in how many of the 16 lines of the truth table?
a) 3 b) 9 c) 10 d) 12
c) 10. 4 entries for $r \wedge s$, 6 more for $p \leftrightarrow q$

b. If assuming P we prove Q, and assuming Q we prove P, then we can derive $P \wedge Q$.
False, this is circular reasoning, they could both be false

c. Consider the propositions p: "It is cloudy", and q: "it is raining".
Which of the following expresses the sentence "It is not cloudy unless it is raining"?
a) $\neg p \wedge q$ b) $\neg p \vee q$ c) $\neg q \vee p$ d) $\neg p \vee \neg q$
b) "if it is not raining, it is not cloudy" $\neg q \rightarrow \neg p = q \vee \neg p$

d. The statement $\forall x : \exists y : \forall z (P(x, y) \wedge R(w, z) \wedge Q(z))$ is a predicate over how many free variables?
a) 1 b) 2 c) 3 d) 4 e) 5
a) one free variable, w

e. The statements $(\exists x : P(x)) \vee (\exists x : Q(x))$ and $\exists x : \exists y : (P(x) \vee Q(y))$ are not equivalent.
False, they are equivalent, check using instantiation

f. If $(\forall x : \exists y : P(x, y)) \rightarrow (\exists x : \forall y : P(x, y))$ then there exists a value a such that $P(a, y)$ holds for all y.
False, the implication could be vacuously true because the premise is false.

g. $(\exists x : \forall y : P(x, y)) \rightarrow (\exists y : \forall x : P(y, x))$ is a tautology.
True, the premise and conclusion are the same, modulo variable renaming

h. The number of binary relations that can be constructed on a set with two elements is
a) 4 b) 8 c) 16 d) 64
c) $16 = 2^4$: there are $4 = 2^2$ pairs, of which each may or may not be in the relation.

i. Consider any surjective function $f : A \rightarrow B$. If f is not injective, there are no injective functions from A to B.
True, if it's not injective, then $|A| > |B|$ by the pigeonhole principle

j. Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be polynomials with natural coefficients. Then $f \circ g = g \circ f$.
False, take $f(x) = x + 1$ and $g(x) = x^2$

k. A binary relation is *well-defined* if and only if its inverse is *one-to-one*.
True, examine the definition

l. Let R and S be two symmetric relations on a set A. The relation $U(x, y) = R(x, y) \vee S(x, y)$ may fail to be symmetric.
False, all edges are bidirectional

m. A partial order on a set A is also a total relation on $A \times A$.
True, it's reflexive, thus total (every element has at least itself)

n. Which of the following relations over the set of all people is an equivalence relation?
a) $\{(x, y) \mid x$ and y are siblings $\}$
b) $\{(x, y) \mid x$ and y are married $\}$
c) $\{(x, y) \mid x$ and y have the same age $\}$
d) none of the three
c) have the same age. a) and b) are not reflexive

o. The relation on naturals defined by "x and y have some common divisor" is an equivalence relation.
True, all naturals have 1 as common divisor. Excluding 1, it is not transitive, try 14, 10, 15

Question 4 (25):

Consider the following compound proposition:
$(r \to u) \land ((p \to s) \to (q \lor r)) \land \neg(p \land r) \land (u \to (p \lor s)) \land ((u \lor p) \to \neg q) \land (r \lor (s \land u))$

a. (5p) Convert it to conjunctive normal form (a conjunction of clauses, which are disjunctions of propositions or their negations).

The structure is already a conjunction, we have to convert the conjuncts which are not clauses:

(1) $r \to u = \neg r \lor u$
(2) $(p \to s) \to (q \lor r) = \neg(p \to s) \lor q \lor r = (p \land \neg s) \lor q \lor r = (p \lor q \lor r) \land (\neg s \lor q \lor r)$
(3) $\neg(p \land r) = \neg p \lor \neg r$
(4) $u \to (p \lor s) = \neg u \lor p \lor s$
(5) $(u \lor p) \to \neg q = \neg(u \lor p) \lor \neg q = (\neg u \land \neg p) \lor \neg q = (\neg u \lor \neg q) \land (\neg p \lor \neg q)$
(6) $r \lor (s \land u) = (r \lor s) \land (r \lor u)$

Putting all together, we obtain

$$(\neg r \lor u) \land (p \lor q \lor r) \land (\neg s \lor q \lor r) \land (\neg p \lor \neg r) \land (\neg u \lor p \lor s) \land (\neg u \lor \neg q) \land (\neg p \lor \neg q) \land (r \lor s) \land (r \lor u)$$

b. (10p) Find an assignment to atomic propositions that makes it true, or show there is none.

The first (1) and last clause (6.2) give $(\neg r \lor u) \land (r \lor u) = (\neg r \lor r) \land u = u$. Thus, $u = \mathrm{T}$.
From (5.1), we get $q = \mathrm{F}$. This also satisfies (5.2).
From (2.2) we get $\neg s \lor r$.
With (6.1) we get $(\neg s \lor r) \land (s \lor r) = (\neg s \lor s) \land r = r$, thus $r = \mathrm{T}$. This satisfies (2).
From (3), we get $p = \mathrm{F}$.
From (4), we get $s = \mathrm{T}$.
All clauses are satisfied. $p = q = \mathrm{F}$, $r = s = u = \mathrm{T}$ is a satisfying assignment (the only one).

c. (10p) Consider the sets $D = (A \,\Delta\, B) \cap C$ and $E = (B \cap (A \cup C)) \setminus (A \cap C)$. Use propositional rules (equational proofs) or set identities to express $D \setminus E$ and $D \cap E$ in the simplest way using A, B, C, and their complements $\overline{A}$, $\overline{B}$, $\overline{C}$.

It's easiest to get the intuition from a Venn diagram and then use set identities. We rewrite:

$D = (A \,\Delta\, B) \cap C = ((A \setminus B) \cup (B \setminus A)) \cap C = ((A \cap \overline{B}) \cup (B \cap \overline{A})) \cap C = (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$.

$E = (B \cap (A \cup C)) \setminus (A \cap C) = B \cap (A \cup C) \cap \overline{A \cap C} = B \cap ((A \cup C) \setminus (A \cap C)) = B \cap (A \,\Delta\, C) = B \cap ((A \cap \overline{C}) \cup (\overline{A} \cap C)) = (A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C)$.

Thus, we have $D = P \cup R$ and $E = Q \cup R$, with $P = A \cap \overline{B} \cap C$, $Q = A \cap B \cap \overline{C}$, $R = \overline{A} \cap B \cap C$.

P, Q and R are pairwise disjoint, as one can also see from a Venn diagram.
$P \cap Q = \emptyset$, $P \cap R = \emptyset$, since $P \subseteq \overline{B}$, and $Q, R \subseteq B$. $Q \cap R = \emptyset$, since $Q \subseteq A, R \subseteq \overline{A}$. Thus:

$D \setminus E = (P \cup R) \setminus (Q \cup R) = P \setminus (Q \cup R) = P = A \cap \overline{B} \cap C$, or, equivalently, $(A \cap C) \setminus B$.

$D \cap E = (P \cup R) \cap (Q \cup R) = (P \cap Q) \cup (P \cap R) \cup (R \cap Q) \cup (R \cap R) = R = \overline{A} \cap B \cap C$, or, equivalently, $(B \cap C) \setminus A$.