Geometric Representations of Graphs

László Lovász
Microsoft Research
One Microsoft Way, Redmond, WA 98052
e-mail: lovasz@microsoft.com

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Part I

Background

Chapter 1

Eigenvalues of graphs

1.1 Matrices associated with graphs

We introduce the adjacency matrix, the Laplacian and the transition matrix of the random walk, and their eigenvalues.

Let G be a (finite, undirected, simple) graph with node set $V(G) = \{1, \ldots, n\}$. The *adjacency matrix* of G is be defined as the $n \times n$ matrix $A_G = (A_{ij})$ in which

$$A_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

We can extend this definition to the case when G has multiple edges: we just let A_{ij} be the number of edges connecting i and j. We can also have weights on the edges, in which case we let A_{ij} be the weight of the edges. We could also allow loops and include this information in the diagonal, but we don't need this in this course.

The *Laplacian* of the graph is defined as the $n \times n$ matrix $L_G = (L_{ij})$ in which

$$L_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -A_{ij}, & \text{if } i \neq j. \end{cases}$$

Here d_i denotes the degree of node i. So $L_G = D_G - A_G$, where D_G is the diagonal matrix of the degrees of G.

In the weighted case, d_i is the sum of weights of edges incident with node i.

The *transition matrix of the random walk on* G is defined as the $n \times n$ matrix $P_G = (P_{ij})$ in which

$$P_{ij} = \frac{1}{d_i} A_{ij}.$$

So $P_G = D_G^{-1} A$.

The matrices A_G and L_G are symmetric, so their eigenvalues are real. The matrix P_G is not symmetric, but it is conjugate to a symmetric matrix. Let

$$N_G = D_G^{-1/2} A_G D_G^{-1/2},$$

then N_G is symmetric, and

$$P_G = D_G^{-1/2} N_G D_G^{1/2}.$$

The matrices P_G and N_G have the same eigenvalues, and so all eigenvalues of P_G are real.

Example 1.1.1 Compute the spectrum of complete graphs, cubes, stars, paths.

We'll often use the (generally non-square) *incidence matrix* of G. This notion comes in two flavors. Let $V(G) = \{1, \dots, n\}$ and $E(G) = \{e_1, \dots, e_m$, and let B_G denote the $n \times m$ matrix for which

$$(B_G)_{ij} = \begin{cases} 1 & \text{if } i \text{ is and endpoint of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Often, however, the following matrix is more useful: Let us fix an orientation of each edge, to get an oriented graph $\overrightarrow{G}$. Then let $B_{\overrightarrow{G}}$ denote the $n \times m$ matrix for which

$$(B_{\overrightarrow{G}})_{ij} = \begin{cases} 1 & \text{if } i \text{ is the head of } e_j, \\ -1 & \text{if } i \text{ is the tail of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Changing the orientation only means scaling some columns by -1, which often does not matter much. For example, it is easy to check that independently of the orientation,

$$L_G = B_{\overrightarrow{G}} B_{\overrightarrow{G}}^{\mathsf{T}}. \tag{1.1}$$

It is worth while to express this equation in terms of quadratic forms:

$$x^{\mathsf{T}} L_G x = \sum_{ij \in E(G)} (x_i - x_j)^2. \tag{1.2}$$

1.2 The largest eigenvalue

1.2.1 Adjacency matrix

The Perron–Frobenius Theorem implies immediately that if G is connected, then the largest eigenvalue $\lambda_{\max}$ of A_G of A_G has multiplicity 1. This eigenvalue is relatively uninteresting, it is a kind of "average degree". More precisely, let $d_{\min}$ denote the minimum degree of G, let $\overline{d}$ be the average degree, and let $d_{\max}$ be the maximum degree.

Proposition 1.2.1 *For every graph G,*

$$\max\{d_{\min}, \sqrt{d_{\max}}\} \le \lambda_{\max} \le \overline{d}.$$

Proof.

□

Example 1.2.2 Compute the largest eigenvalue of a star.

1.2.2 Laplacian

For the Laplacian L_G, this corresponds to the smallest eigenvalue, which is really uninteresting, since it is 0:

Proposition 1.2.3 *The Laplacian L_G is singular and positive semidefinite.*

Proof. The proof follows immediately from (1.1) or (1.2), which show that L_G is positive semidefinite. Since $\mathbf{1} = (1, \ldots, 1)^\mathsf{T}$ is in the null space of L_G, it is singular. □

If G is connected, then 0, as an eigenvalue of L_G, has multiplicity 1; we get this by applying the Perron–Frobenius Theorem to $cI - L_G$, where c is a large real number. The eigenvector belonging to this eigenvalue is $\mathbf{1} = (1, \ldots, 1)^\mathsf{T}$ (and its scalar multiples).

We note that for a general graph, the multiplicity of the 0 eigenvalue of the Laplacian is equal to the number of connected components. Similar statement is not true for the adjacency matrix (if the largest eigenvalues of the connected components of G are different, then the largest eigenvalue of the whole graph has multiplicity 1). This illustrates the phenomenon that the Laplacian is often better behaved algebraically than the adjacency matrix.

1.2.3 Transition matrix

The largest eigenvalue of P_G is 1, and it has multiplicity 1 for connected graphs. It is straightforward to check that the right eigenvector belonging to it is $\mathbf{1}$, and the left eigenvector is given by $\pi_i = d_i/(2m)$ (where m is the number of edges). This vector π describes the *stationary distribution* of a random walk, and it is very important in the theory of random walks (see later).

1.3 The smallest eigenvalue

Proposition 1.3.1 (a) *A graph is bipartite if and only if its spectrum is symmetric about the origin.*

(b) A connected graph G is bipartite if and only if $\lambda_{\min}(G) = -\lambda_{\max}(G)$.

Proof.

□

The "only if" part of Proposition 1.3.1 can be generalized: The ratio between the largest and smallest eigenvalue can be used to estimate the chromatic number [68].

Theorem 1.3.2

$$\chi(G) \geq 1 + \frac{\lambda_{\min}}{\lambda_{\max}}.$$

Proof. Let $k = \chi(G)$, then A_G can be partitioned as

$$\begin{pmatrix} 0 & M_{12} & \ldots & M_{1k} \\ M_{21} & 0 & & M_{2k} \\ \vdots & \vdots & \ddots & \\ M_{k1} & M_{k2} & & 0, \end{pmatrix}$$

where M_{ij} is an $m_i \times m_j$ matrix (where m_i is the number of points with color i).

Let $\mathbf{v}$ be an eigenvector belonging to λ_1. Let us break $\mathbf{v}$ into pieces $\mathbf{v}_1, \dots, \mathbf{v}_k$ of length $m_1, \dots, m_k$, respectively. Set

$$\mathbf{w}_i = \begin{pmatrix} |\mathbf{v}_i| \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{m_i} \quad \mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_k \end{pmatrix}.$$

Let B_i be any orthogonal matrix such that

$$B_i \mathbf{w}_i = \mathbf{v}_i \qquad (i = 1, \dots, k),$$

and

$$B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}.$$

Then $B\mathbf{w} = \mathbf{v}$ and

$$B^{-1}AB\mathbf{w} = B^{-1}A\mathbf{v} = \lambda_1 B^{-1}\mathbf{v} = \lambda_1 \mathbf{w}$$

so $\mathbf{w}$ is an eigenvector of $B^{-1}AB$. Moreover, $B^{-1}AB$ has the form

$$\begin{pmatrix} 0 & B_1^{-1}A_{12}B_2 & \dots & B_1^{-1}A_{1k}B_k \\ B_2^{-1}A_{21}B_1 & 0 & & B_2^{-1}A_{2k}B_k \\ \vdots & & \ddots & \vdots \\ B_k^{-1}A_{k1}B_1 & B_k^{-1}A_{k2}B_2 & \dots & 0 \end{pmatrix}.$$

Pick the entry in the upper left corner of each of the k^2 submatrices $B_i^{-1}A_{ij}B_j$ ($A_{ii} = 0$), these form a $k \times k$ submatrix D. Observe that

$$\mathbf{u} = \begin{pmatrix} |\mathbf{v}_1| \\ \vdots \\ |\mathbf{v}_k| \end{pmatrix}$$

is an eigenvector of D; for $\mathbf{w}$ is an eigenvector of $B^{-1}AB$ and has 0 entries on places corresponding to those rows and columns of $B^{-1}AB$, which are to be deleted to get D. Moreover, the eigenvalue belonging to $\mathbf{u}$ is λ_1.

Let $\alpha_1 \geq \dots \geq \alpha_k$ be the eigenvalues of D. Since D has 0's in its main diagonal,

$$\alpha_1 + \dots + \alpha_k = 0.$$

On the other hand, λ_1 is an eigenvalue of D and so

$$\lambda_1 \leq \alpha_1,$$

while by the Interlacing Eigenvalue Theorem

$$\lambda_n \leq \alpha_k, \dots, \lambda_{n-k+2} \leq \alpha_2.$$

Thus

$$\lambda_n + \cdots + \lambda_{n-k+2} \le \alpha_k + \cdots + \alpha_2 = -\alpha_1 \le -\lambda_1.$$

□

Remark 1.3.3 The proof did not use that the edges were represented by the number 1, only that the non-edges and diagonal entries were 0. So if we want to get the strongest possible lower bound on the chromatic number that this method provides, we can try to find a way of choosing the entries in A corresponding to edges of G in such a way that the right hand side is minimized. This can be done efficiently, and this idea will be important in Chapter 7.

The smallest eigenvalue can be used to characterize linegraphs [67].

Proposition 1.3.4 *Let H be the linegraph of G, and let $\lambda_{\min}$ be the smallest eigenvalue of H. Then $\lambda_{\min} \ge -2$; if $|E(G)| > |V(G)|$, then $\lambda_{\min} = -2$.*

Proof. It is easy to check that we have

$$A_{L(G)} = B_G^{\mathsf{T}} B_G - 2I$$

follows easily.

Since $B_G^T B_G$ is positive semidefinite, all of its eigenvalues are non-negative. Hence, the eigenvalues of $A_{L(G)}$ are ≥ -2. Moreover, if $|V(G)| < |E(G)|$, then

$$r(B^T B) = r(B) \le |V(G)| < |E(G)|$$

($r(X)$ is the rank of the matrix X). So, $B^T B$ has at least one 0 eigenvalue, i.e. $A_{L(G)}$ has at least one -2 eigenvalue. □

1.4 The eigenvalue gap

The gap between the second and the first eigenvalues is an extremely important parameter in many branches of mathematics.

If the graph is connected, then the largest eigenvalue of the adjacency matrix as well as the smallest eigenvalue of the Laplacian have multiplicity 1. We can expect that the gap between this and the nearest eigenvalue is related to some kind of connectivity measure of the graph. Indeed, fundamental results due to Alon–Milman [3], Alon [2] and Jerrum–Sinclair [72] relate the eigenvalue gap to expansion (isoperimetric) properties of graphs. These results can be considered as discrete analogues of Cheeger's inequality in differential geometry.

There are many related (but not equivalent) versions of these results. We illustrate this connection by two versions that are of special interest: a spectral characterization of expanders and a bound on the mixing time of random walks on graphs. For this, we discuss very briefly expanders and also random walks and their connections with eigenvalues (see [1] and [94] for more).

The multiplicity of the second largest eigenvalue will be discussed in connection with the Colin de Verdière number (Chapter 8).

1.4.1 Expanders

An *expander* is a regular graph with small degree in which the number of neighbors of any set containing at most half of the nodes is at least a constant factor of its size. To be precise, an ε-expander is a graph $G = (V, E)$ in which for every set $S \subset V$ with $|S| \leq |V|/2$, the number of nodes in $V \setminus S$ adjacent to some node in S is at least $\varepsilon|S|$.

Expanders play an important role in many applications of graph theory, in particular in computer science. The most important expanders are d-regular expanders, where $d \geq 3$ is a small constant. Such graphs are not easy to construct. One method is to do a random construction: for example, we can pick d random perfect matchings on $2n$ nodes (independently, uniformly over all perfect matchings), and let G be the union of them. Then a moderately complicated computation shows that G is an ε-expander with positive probability for a sufficiently small ε. Deterministic constructions are much more difficult to obtain; the first construction was found by Margulis [97]; see also [95]. Most of these constructions are based on deep algebraic facts.

Our goal here is to state and prove a spectral characterization of expanders, due to Alon [2], which plays an important role in analyzing some of the above mentioned algebraic constructions. note that since we are considering only regular graphs, the adjacency matrix, the Laplacian and the transition matrix are easily expressed, and so we shall only consider the adjacency matrix.

Theorem 1.4.1 *Let G be a d-regular graph.*

(a) *If $d - \lambda_2 \geq 2\varepsilon d$, then G is an ε-expander.*

(b) *If G is an ε-expander, then $d - \lambda_2 \geq \varepsilon^2/5$.*

Proof. The proof is similar to the proof of Theorem 1.4.5 below. □

1.4.2 Random walks

A *random walk* on a graph G is a random sequence $(v^0, v^1, \dots)$ of nodes constructed as follows: We pick a starting point v^0 from a specified initial distribution σ, we select a neighbor v^1 of it at random (each neighbor is selected with the same probability $1/d(v^0)$), then we select a neighbor v^2 of this node v^1 at random, etc. We denote by σ^k the distribution of v^k.

In the language of probability theory, a random walk is a finite time-reversible Markov chain. (There is not much difference between the theory of random walks on graphs and the theory of finite Markov chains; every Markov chain can be viewed as random walk on a directed graph, if we allow weighted edges, and every time-reversible Markov chain can be viewed as random walks on an edge-weighted undirected graph.)

Let π denote the probability distribution in which the probability of a node is proportional to its degree:

$$\pi(v) = \frac{d(v)}{2m}.$$

This distribution is called the *stationary distribution* of the random walk. It is easy to check that if v^0 is selected from π, then after any number of steps, v^k will have the same distribution π. This explains the name of π. Algebraically, this means that π is a left eigenvector of P_G with eigenvalue 1:

$$\pi^T P_G = \pi^T.$$

Theorem 1.4.2 *If G is a connected nonbipartite graph, then $\sigma^k \to \pi$ for every starting distribution σ.*

It is clear that the conditions are necessary.

Before proving this theorem, let us make some remarks on one of its important applications, namely *sampling*. Suppose that we want to pick a random element uniformly from some finite set. We can then construct a connected nonbipartite regular graph on this set, and start a random walk on this graph. A node of the random walk after sufficiently many steps is therefore essentially uniformly distributed.

(It is perhaps surprising that there is any need for a non-trivial way of generating an element from such a simple distribution as the uniform. But think of the first application of random walk techniques in real world, namely shuffling a deck of cards, as generating a random permutation of 52 elements from the uniform distribution over all permutations. The problem is that the set we want a random element from is exponentially large. In many applications, it has in addition a complicated structure; say, we consider the set of lattice points in a convex body or the set of linear extensions of a partial order. Very often this random walk sampling is the only known method.)

With this application in mind, we see that not only the fact of convergence matters, but also the rate of this convergence, called the *mixing rate*. The proof below will show how this relates to the eigenvalue gap. In fact, we prove:

Theorem 1.4.3 *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of P_G, and let $\mu = \max\{\lambda_2, \lambda_n\}$. Then for every starting node i and any node j, and every $t \geq 0$, we have*

$$|\Pr(v^t = j) - \pi(j)| \leq \sqrt{\frac{\pi(j)}{\pi(i)}}\mu^t.$$

More generally, for every set $A \subseteq V$,

$$|\Pr(v^t \in A) - \pi(A)| \leq \sqrt{\frac{\pi(A)}{\pi(i)}}\mu^t.$$

Proof. We prove the first inequality; the second is left to the reader as an exercise. We know that the matrix N_G has the same eigenvalues as P_G, and it is symmetric, so we can write it as

$$N_G = \sum_{k=1}^{n} \lambda_k v_k v_k^{\mathsf{T}},$$

where $v_1, \ldots, v_n$ are mutually orthogonal eigenvectors. It is easy to check that we can choose

$$v_{1i} = \sqrt{\pi_i}$$

(we don't know anything special about the other eigenvectors). Hence we get

$$\begin{aligned}\Pr(v^t = j) = (P^t)_{ij} = e_i^{\mathsf{T}} D^{-1/2} N^t d^{1/2} e_j &= \sum_{k=1}^{n} \lambda_k^t (e_i^{\mathsf{T}} D^{-1/2} v_k)(e_j^{\mathsf{T}} D^{1/2} v_k) \\ &= \sum_{k=1}^{n} \lambda_k^t \frac{1}{\sqrt{\pi(i)}} v_{ki} \sqrt{\pi(j)} v_{kj} = \pi(j) + \sqrt{\frac{\pi(j)}{\pi(i)}} \sum_{k=2}^{n} \lambda_k^t v_{ki} v_{kj}.\end{aligned}$$

Here the first term is the limit; we need to estimate the second. We have

$$\left|\sum_{k=2}^{n} \lambda_k^t v_{ki} v_{kj}\right| \leq \mu^t \sum_{k=2}^{n} \left|v_{ki} v_{kj}\right| \leq \mu^t \sum_{k=1}^{n} \left|v_{ki} v_{kj}\right| \leq \mu^t \left(\sum_{k=1}^{n} v_{ki}^2\right)^{1/2} \left(v_{kj}^2\right)^{1/2} = \mu^t.$$

This proves the inequality. □

If we want to find a bound on the number of steps we need before, say,

$$|\Pr(v^k \in A) - \pi(A)| < \varepsilon$$

holds for every j, then it suffices to find a k for which

$$\mu^k < \varepsilon\sqrt{\pi_i}.$$

Writing $mu = 1 - \gamma$, and using that $1 - \gamma < e^{-\gamma}$, it suffices to have

$$e^{-\gamma k} < \varepsilon\sqrt{\pi_i},$$

and expressing k,

$$k > \frac{1}{\gamma}\Big(\ln\frac{1}{\varepsilon} + \frac{1}{2}\ln\frac{1}{\pi_i}\Big)$$

So we see that (up to logarithmic factors), it is the reciprocal of the eigenvalue gap that governs the mixing time.

In applications, the appearance of the smallest eigenvalue λ_n is usually not important, and what we need to work on is bounding the *eigenvalue gap* $1 - \lambda_2$. The trick is the following: If the smallest eigenvalue is too small, then we can modify the walk as follows. At each step, we flip a coin and move with probability $1/2$ and stay where we are with probability $1/2$. The stationary distribution of this modified walk is the same, and the transition matrix P_G is replaced by $\frac{1}{2}(P_G + I)$. For this modified walk, all eigenvalues are nonnegative, and the eigenvalue gap is half of the original. So applying the theorem to this, we only use a factor of 2.

The eigenvalues of a general graph are usually difficult to compute; we would like to bound them by more combinatorial quantities. Let $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of P_G.

Lemma 1.4.4 *For every graph G we have*

$$1 - \lambda_2 = \frac{1}{2m}\min\sum_{(i,j)\in E}(x_i - x_j)^2,$$

where the minimum is taken over all vectors $x \in \mathbb{R}^V$ such that

$$\sum_{i\in V}\pi_i x_i = 0, \qquad \sum_{i\in V}\pi_i x_i^2 = 1.$$

Proof. As remarked before, the symmetrized matrix $N_G = D_G^{1/2} P_G D_G^{-1/2}$ has the same eigenvalues as P_G. For a symmetric matrix, the second largest eigenvalue can be obtained as

$$\lambda_2 = \max y^{\mathsf{T}} N_G y,$$

where y ranges over all vectors of unit length orthogonal to the eigenvector belonging to the largest eigenvalue. This latter eigenvector is given by $v_i = \sqrt{\pi_i}$, so the conditions of y are

$$\sum_{i\in V}\sqrt{\pi_i}y_i = 0, \qquad \sum_{i\in V} y_i^2 = 1.$$

Let us introduce $x_i = y_i/\sqrt{\pi_i}$, then the conditions on x are as in the lemma. Furthermore,

$$1 - y^{\mathsf{T}} N_G y = \sum_{i \in V} \pi_i x_i^2 - 2 \sum_{ij \in E} \frac{1}{\sqrt{d_i d_j}} \sqrt{\pi_i \pi_j} x_i x_j$$
$$= \frac{1}{2m} \sum_{i \in V} d_i x_i^2 - 2 \sum_{ij \in E} x_i x_j$$
$$= \sum_{(i,j) \in E} (x_i - x_j)^2,$$

which proves the Lemma. □

The *conductance* of a graph $G = (V, E)$ is defined as follows. For two sets $S_1, S_2 \subseteq V$, let $e_G(S_1, S_2)$ denote the number of edges ij with $i \in S_1, j \in S_2$. We also set $Q(S_1, S_2) = e_G(S_1, S_2)/(2m)$. We define

$$\Phi(G) = \min_{\emptyset \subset S \subset V} \frac{Q(S, V \setminus S)}{\pi(S)\pi(V \setminus S)}.$$

(Explanation: In a stationary random walk on G, we cross every edge in every direction with the same frequency, once in every $2m$ steps on the average. So $Q(S, V \setminus S)$ is the frequency with which we step out from S. If instead we consider a sequence of independent samples from π, the frequency with which we step out from S is $\pi(S)\pi(V \setminus S)$. The ratio of these two frequencies is one of many possible ways comparing a random walk with a sequence of independent samples.)

The following basic inequality was proved by Jerrum and Sinclair [72]:

Theorem 1.4.5 *For every graph G,*

$$\frac{\Phi(G)^2}{8} \leq 1 - \lambda_2 \leq \Phi(G)$$

Proof. The upper bound is easy: let $\emptyset \neq S \subset V$ be a set with

$$\frac{Q(S, V \setminus S)}{\pi(S)\pi(V \setminus S)} = \Phi(G).$$

Let x be a vector on the nodes defined by

$$x_i = \begin{cases} \sqrt{\frac{\pi(V \setminus S)}{\pi(S)}} & \text{if } i \in S, \\ -\sqrt{\frac{\pi(S)}{\pi(V \setminus S)}} & \text{if } i \in V \setminus S. \end{cases}$$

It is easy to check that

$$\sum_{i \in V} \pi_i x_i = 0, \qquad \sum_{i \in V} \pi_i x_i^2 = 1.$$

Thus by Lemma 1.4.4,

$$1 - \lambda_2 \geq \frac{1}{2m} x^{\mathsf{T}} A_G x = \frac{1}{2m} \sum_{ij \in E} (x_i - x_j)^2$$
$$= \frac{1}{2m} e_G(S, V \setminus S) \left(\sqrt{\frac{\pi(V \setminus S)}{\pi(S)}} + \sqrt{\frac{\pi(S)}{\pi(V \setminus S)}} \right)^2 = \frac{1}{2m} \frac{e_G(S, V \setminus S)}{\pi(S)\pi(V \setminus S)}$$
$$= \Phi(G).$$

To prove the lower bound, we need a lemma. For every real vector $y = (y_1, \ldots, y_n)$, we define its *median* as the $\lfloor n/2 \rfloor^{\text{th}}$ largest entry of y.

Lemma 1.4.6 *Let $G = (V, E)$ be a graph with conductance $\Phi(G)$. Let $y \in \mathbb{R}^V$, and let $\overline{y}$ be the median of y. Then*

$$\sum_{(i,j)\in E} |y_i - y_j| \geq \Phi \sum_i |y_i - \overline{y}|.$$

Proof. [of the Lemma] We may label the nodes so that $y_1 \leq y_2 \leq \ldots \leq y_n$. We also may assume that $\overline{y} = 0$ (the assertion of the Lemma is invariant under shifting the entries of y). Substituting

$$y_j - y_i = (y_{i+1} - y_i) + \cdots + (y_j - y_{j-1}),$$

we have

$$\sum_{(i,j)\in E} |y_i - y_j| = \sum_{i=1}^{n-1} e(S_i, V \setminus S_i)(y_{i+1} - y_i).$$

By the definition of Φ, this implies

$$\begin{aligned}\sum_{(i,j)\in E} |y_i - y_j| &\geq \Phi \sum_{i=1}^{n-1} \min\{i, n-i\}(y_{i+1} - y_i) = \Phi \left(\sum_{i \leq n/2} y_i - \sum_{i > n/2} y_i \right) \\ &= \Phi \sum_i |y_i|.\end{aligned}$$

□

Now we return to the proof of the lower bound in Theorem 1.4.5. Let x be a unit length eigenvector belonging to λ_2. We may assume that the nodes are labeled so that $x_1 \geq x_2 \geq \ldots \geq x_n$. Let $\overline{x}$ be the median of x. Setting $z_i = \big(\max\{0, x_i - \overline{x}\}\big)$ and choosing the sign of x appropriately, we may assume that

$$\sum_i z_i^2 \geq \frac{1}{2} \sum_i (x_i - \overline{x})^2 = \frac{1}{2} \sum_i x_i^2 - \overline{x} \sum_i x_i + \frac{n}{2} \overline{x}^2 = \frac{1}{2} + \frac{n}{2} \overline{x}^2 \geq \frac{1}{2}.$$

By Lemma 1.4.6

$$\sum_{(i,j)\in E} |z_i^2 - z_j^2| \geq \Phi \sum_i z_i^2.$$

On the other hand, using the Cauchy-Schwartz inequality,

$$\sum_{(i,j)\in E} |z_i^2 - z_j^2| \leq \left(\sum_{(i,j)\in E} (z_i - z_j)^2 \right)^{1/2} \left(\sum_{(i,j)\in E} (z_i + z_j)^2 \right)^{1/2}.$$

Here the second factor can be estimated as follows:

$$\sum_{(i,j)\in E} (z_i + z_j)^2 \leq 2 \sum_{(i,j)\in E} (z_i^2 + z_j^2) = 2d \sum_i z_i^2.$$

Combining these inequalities, we obtain

$$\sum_{(i,j)\in E}(z_i-z_j)^2 \geq \left(\sum_{(i,j)\in E}|z_i^2-z_j^2|\right)^2 \Big/ \sum_{(i,j)\in E}(z_i+z_j)^2$$
$$\geq \Phi^2\left(\sum_i z_i^2\right)^2 \Big/ 2d\sum_i z_i^2 = \frac{\Phi^2}{2d}\sum_i z_i^2 \geq \frac{\Phi^2}{4d}.$$

Since

$$\sum_{(i,j)\in E}(x_i-x_j)^2 \geq \sum_{(i,j)\in E}(z_i-z_j)^2,$$

from here we can conclude by Lemma 1.4.4. □

The quantity $\Phi(G)$ is NP-complete to compute. An important theorem of Leighton and Rao gives an approximate min-max theorem for it, which also yields a polynomial time approximation algorithm. This will be discussed in Chapter 10.

1.5 The number of different eigenvalues

Multiplicity of eigenvalues usually corresponds to symmetries in the graph (although the correspondence is not exact). We prove two results in this direction. The following theorem was proved by Mowshowitz [103] and Sachs [115]:

Theorem 1.5.1 *If all eigenvalues of A are different, then every automorphism of A has order 1 or 2.*

Proof. Every automorphism of G can be described by a permutation matrix P such that $AP = PA$. Let u be an eigenvector of A with eigenvalue λ. Then

$$A(Pu) = PAu = P(\lambda u) = \lambda(Pu),$$

so Pu is also an eigenvector of A with the same eigenvalue. Since Pu has the same length as u, it follows that $Pu = \pm u$ and hence $P^2u = u$. This holds for every eigenvector u of A, and since there is a basis consisting of eigenvectors, it follows that $P^2 = I$. □

A graph G is called *strongly regular*, if it is regular, and there are two nonnegative integers a and b such that for every pair i, j of nodes the number of common neighbors of i and j is

$$\begin{cases} a, & \text{if } a \text{ and } b \text{ are adjacent,} \\ b, & \text{if } a \text{ and } b \text{ are nonadjacent.} \end{cases}$$

Example 1.5.2 Compute the spectrum of the Petersen graph, Paley graphs, incidence graphs of finite projective planes.

The following characterization of strongly regular graphs is easy to prove:

Proposition 1.5.3 *A connected graph G is strongly regular if and only if it is regular and A_G has at most 3 different eigenvalues.*

Proof. The adjacency matrix of a strongly regular graph satisfies

$$A^2 = aA + b(J - A - I) + dI. \tag{1.3}$$

The largest eigenvalue is d, all the others are roots of the equation

$$\lambda^2 - (a-b)\lambda - (d-b), \tag{1.4}$$

Thus there are at most three distinct eigenvalues.

Conversely, suppose that G is d-regular and has at most three different eigenvalues. One of these is d, with eigenvector $\mathbf{1}$. Let λ_1 and λ_2 be the other two (I suppose there are two more—the case when there is at most one other is easy). Then

$$B = A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I$$

is a matrix for which $Bu = 0$ for every eigenvector of A except $\mathbf{1}$ (and its scalar multiples). Furthermore, $B\mathbf{1} = c\mathbf{1}$, where $c = (d-\lambda_1)(d-\lambda_2)$. Hence $B = (c/n)J$, and so

$$A^2 = (\lambda_1 + \lambda_2)A - \lambda_1\lambda_2 I + (c/n)J.$$

This means that $(A^2)_{ij}$ $(i \neq j)$ depends only on whether i and j are adjacent, proving that G is strongly regular. □

Example 1.5.4 Describe all disconnected strongly regular graphs. Show that there are disconnected graphs with only 3 distinct eigenvalues that are not strongly regular.

We can get more out of equation (1.4). We can solve it:

$$\lambda_{1,2} = \frac{a - b \pm \sqrt{(a-b)^2 + 4(d-b)}}{2}. \tag{1.5}$$

Counting induced paths of length 2, we also get the equation

$$(d - a - 1)d = (n - d - 1)b. \tag{1.6}$$

Let m_1 and m_2 be the multiplicities of the eigenvalues λ_1 and λ_2. Clearly

$$m_1 + m_2 = n - 1 \tag{1.7}$$

Taking the trace of A, we get

$$d + m_1\lambda_1 + m_2\lambda_2 = 0,$$

or

$$2d + (n-1)(a-b) + (m_1 - m_2)\sqrt{(a-b)^2 + 4(d-b)} = 0. \tag{1.8}$$

If the square root is irrational, the only solution is $d = (n-1)/2$, $b = (n-1)/4$, $a = b-1$. There are many solutions where the square root is an integer.

A nice application of these formulas is the "Friendship Theorem":

Theorem 1.5.5 *If G is a graph in which every two nodes have exactly one common neighbor, then it has a node adjacent to every other node.*

Proof. First we show that two non-adjacent nodes must have the same degree. Suppose that there are two non-adjacent nodes u, v of different degree. For every neighbor w of u there is a common neighbor w' of w and v. For different neighbors w_1 and w_2 of u, the nodes w_1' and w_2' must be different, else $w-1$ and $w-2$ would have two common neighbors. So v has at least as many neighbors as u. By a symmetric reasoning, we get $d_u = d_v$.

If G has a node v whose degree occurs only once, then by the above, v must be connected to every other node, and we are done. So suppose that no such node exists.

If G has two nodes u and v of different degree, then it contains two other nodes x and y such that $d_u = d_x$ and $d_v = d_y$. But then both x and u are common neighbors of v and y, contradicting the assumption.

Now if G is regular, then it is strongly regular, and $a = b = 1$. From (1.8),

$$d + (m_1 - m_2)\sqrt{d-1} = 0.$$

The square root must be integral, hence $d = k^2 + 1$. But then $k \mid k^2 + 1$, whence $k = 1$, $d = 2$, and the graph is a triangle, which is not a counterexample. □

Exercise 1.1 Prove that every graph with two different eigenvalues is complete.

Exercise 1.2 Construct graphs with three different eigenvalues that are not regular (and hence not strongly regular).

1.6 Eigenvectors

Relatively little is known about the eigenvectors belonging to various eigenvalues. For a connected graph, the Perron–Frobenius Theorem implies that the eigenvector belonging to the largest eigenvalue $\lambda_{\max}$ is uniquely determined (up to scaling), and it is all-positive (or all-negative).

For the other eigenvectors, we prove an important lemma of Van der Holst [62] and Colin de Verdière [28].

Lemma 1.6.1 *Let G be a connected graph, let λ be an eigenvalue of A with multiplicity s, and let r be the number of eigenvalues larger than λ. Let x be any eigenvector belonging to λ, and let a, b and c denote the number of connected components of the subgraph spanned by* $\mathrm{supp}_+(x)$, $\mathrm{supp}_-(x)$ *and* $\mathrm{supp}(x)$, *respectively. Then*

(a) $c \le s$;

(b) $a + b \le r + c$;

(c) *if x has minimal support among the eigenvectors belonging to λ, then $a + b \le r + 1$.*

(d) *A has at least $a + b$ eigenvalues $\ge \lambda$.*

Proof. Let $M = \lambda I - A$. Let $H_1, \ldots, H_a$ and $H_{a+1}, \ldots, H_{a+b}$ be the connected components of the subgraph spanned by $\mathrm{supp}_+(x)$ and $\mathrm{supp}_(x)$, respectively. Let x_i be the restriction of x onto H_i, extended by 0's so that it is a vector in $\mathbb{R}^V$. Thus $x = x_1 + \cdots + x_{a+b}$.

For $z \in \mathbb{R}^{a+b}$, let

$$y = \sum_{i=1}^{a+b} z_i x_i. \tag{1.9}$$

Then

$$y^{\mathsf{T}}My = \sum_{i,j=1}^{a+b} z_i z_j x_i^{\mathsf{T}} M x_j = \sum_{i,j=1}^{a+b} W_{ij} z_i z_j = z^{\mathsf{T}} W z,$$

where $W_{ij} = -x_i^{\mathsf{T}} M x_j$ and W is the $(a+b) \times (a+b)$ matrix $W = (W_{ij})$. We can observe the following properties:

(i) $W_{ij} = 0$ if there is no edge between H_i and H_j; in particular, if $1 \le i, j \le a$ or $a+1 \le i, j \le b$.

(ii) $W_{ij} \ge 0$ for $i \ne j$. It suffices to verify this when $1 \le i \le a$ and $a+1 \le j \le b$. But then in $W_{ij} = x_i^{\mathsf{T}} M x_j$ all non-zero terms are positive.

(iii) $\sum_j W_{ij} = x_i^{\mathsf{T}} \sum_j M x_j = x_i^{\mathsf{T}} M x = 0$.

It follows from (iii) that the quadratic form belonging to W can be written as

$$z^{\mathsf{T}} W z = \sum_{i,j=1}^{a+b} W_{ij} z_i z_j = -\sum_{i<j} W_{ij} (z_i - z_j)^2,$$

and so this form is negative semidefinite. Furthermore, $z^{\mathsf{T}} W z = 0$ if and only if $z_i = z_j$ whenever there is an edge between the components H_i and H_j; in other words, z is constant on the connected components of $\mathrm{supp}(x)$. So the dimension of the nullspace of W is exactly c. If $Wz = 0$, then for every vector y defined by (1.9) we have $Mz = 0$, so the dimension of the nullspace of M is at least c, which proves (a). Furthermore, $z^{\mathsf{T}} W z < 0$ on every vector z in the range of W, which has dimension $a+b-c$. So $y^{\mathsf{T}} M y < 0$ on an $(a+b-c)$-dimensional subspace, and hence M must have at least $a+b-c$ negative eigenvalues, proving (b).

Suppose that x has minimal support among the eigenvectors belonging to λ. Then no vector $z \in \mathbb{R}^{a+b}$ with $Wz = 0$ can have a zero coordinate, since then the corresponding y would be an eigenvector of A belonging to λ with smaller support. Hence the dimension of the nullspace of W is at most 1. From here (c) follows.

(d) is trivial by (a) and (b), since $r + s \ge r + c \ge a + b$. □

Exercise 1.3 Let $G = (V, E)$ be a simple graph, and define

$$\rho(G) = \min_{\emptyset \subset S \subset V} \frac{e_G(S, V \setminus S)}{|S| \cdot |V \setminus S|}.$$

Let λ_2 denote the second smallest eigenvalue of the Laplacian L_G of a graph G. Then

$$\lambda_2 \le n\rho(G) \le \sqrt{\lambda_2 d_{\max}}.$$

Chapter 2

Convex polytopes

2.1 Polytopes and polyhedra

The convex hull of a finite set of points in $\mathbb{R}^d$ is called a (convex) *polytope*. The intersection of a finite number of halfspaces in $\mathbb{R}^d$ is called a (convex) *polyhedron*.

Proposition 2.1.1 *Every polytope is a polyhedron. A polyhedron is a polytope if and only if it is bounded.*

For every polytope, there is a unique smallest affine subspace that contains it, called its *affine hull*. The *dimension* of a polytope is the dimension of it affine hull. A polytope in $\mathbb{R}^d$ that has dimension d (equivalently, that has an interior point) is called a *d-polytope*.

A hyperplane H is said to *support* the polytope if it has a point in common with the polytope and the polytope is contained in one of the closed halfspaces with boundary H. A *face* of a polytope is its intersection with a supporting hyperplane. A face of a polytope that has dimension one less than the dimension of the polytope is called a *facet*. A face of dimension 0 (i.e., a single point) is called a *vertex*.

Proposition 2.1.2 *Every face of a polytope is a polytope. Every vertex of a face is a vertex of the polytope. Every polytope has a finite number of faces.*

Proposition 2.1.3 *Every polytope is the convex hull of its facets. The set of vertices is the unique minimal finite set of points whose convex hull is the polytope.*

Let P be a d-polytope. Then every facet F of P spans a (unique) supporting hyperplane, and the hyperplane is the boundary of a uniquely determined halfspace that contains the polytope. We'll call this halfspace the *halfspace of F*.

Proposition 2.1.4 *Every polytope is the intersection of the halfspaces of its facets.*

2.2 The skeleton of a polytope

The vertices and edges of a polytope P form a simple graph G_P, which we call the *skeleton* of the polytope.

Proposition 2.2.1 *Let P be a polytope in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$. Let u and v be vertices of P such that $a^\mathsf{T}u < a^\mathsf{T}v$. Then there is a vertex w of P such that uw is an edge and $a^\mathsf{T}u < a^\mathsf{T}w$.*

Another way of formulating this is that if we consider the linear objective function $a^\mathsf{T}x$ on a polytope P, then from any vertex we can walk on the skeleton to a vertex that maximizes the objective function so that the value of the objective function increases at every step. This important fact is the basis for the *Simplex method*.

For our purposes, however, the following corollaries of Proposition 2.2.1 will be important:

Corollary 2.2.2 *The skeleton of any polytope is a connected graph.*

Corollary 2.2.3 *Let G be the skeleton of a d-polytope, and let H be a halfspace containing an interior point of the polytope. Then the subgraph of G_P induced by those vertices of P that are contained in this halfspace is connected.*

From Corollary 2.2.3, it is not hard to derive

Theorem 2.2.4 *The skeleton of a d-dimensional polytope is d-connected.*

2.3 Polar, blocker and antiblocker

Let P be a convex polytope containing the origin as an interior point. Then the *polar* of P is defined as

$$P^* = \{x \in \mathbb{R}^d : \ x^\mathsf{T}y \leq 1 \forall y \in P\}$$

Proposition 2.3.1 *(a) The polar of a polytope is a polytope. For every polytope P we have $(P^*)^* = P$.*

(b) Let $v_0, \ldots, v_m$ be the vertices of a k-dimensional face F of P. Then

$$F^\perp = \{x \in P^* : \ v_0^\mathsf{T}x = 1, \ldots, v_m^\mathsf{T}x = 1\}$$

defines a $d-k-1$-dimensional face of P^*. Furthermore, $(F^\perp)^\perp = F$.

In particular, every vertex v of P corresponds to a facet $v^\perp$ of P^* and vice versa. The vector v is a normal vector of the facet $v^\perp$.

There are two constructions similar to polarity that concern polyhedra that do not contain the origin in their interior; rather, they are contained in the nonnegative orthant.

A polyhedron P in $\mathbb{R}^d$ is called *ascending*, if $P \subseteq \mathbb{R}^d_+$ and whenever $x \in \mathsf{P}$, $y \in \mathbb{R}^d$ and $y \geq x$ then $y \in P$.

The *blocker* of an ascending polyhedron is defined by

$$P^{\mathrm{bl}} = \{x \in \mathbb{R}^d_+ : \ x^\mathsf{T}y \leq 1 \forall y \in P\}.$$

Proposition 2.3.2 *The blocker of an ascending polyhedron is an ascending polyhedron. For every ascending polyhedron P we have $(P^{\mathrm{bl}})^{\mathrm{bl}} = P$.*

The correspondence between faces of P and P^{bl} is a bit more complicated than for polarity, and we describe the relationship between vertices and facets only. Every vertex v of P gives rise to a facet $v\perp$, which corresponds to the halfspace $v^{\mathsf{T}}x \geq 1$. This construction gives all the facets of P^{bl}, except possibly those corresponding to the nonnegativity constraints $x_i \geq 0$, which may or may not define facets.

A d-polytope P is called a *corner polytope*, if $P \subseteq \mathbb{R}^d_+$ and whenever $x \in \mathsf{P}$, $y \in \mathbb{R}^d$ and $0 \leq y \leq x$ then $y \in P$.

The *antiblocker* of a corner polytope is defined by

$$P^{\mathrm{abl}} = \{x \in \mathbb{R}^d_+ : \ x^{\mathsf{T}}y \leq 1 \forall y \in P\}.$$

Proposition 2.3.3 *The antiblocker of a corner polytope is a corner polytope. For every corner polytope P we have $(P^{\mathrm{abl}})^{\mathrm{abl}} = P$.*

The correspondence between faces of P and P^{abl} is more complicated than for the blocking polyhedra. The nonnegativity constraints $x_i \geq 0$ always define facets, and they don't correspond to vertices in the antiblocker. All other facets of P correspond to vertices of P^{abl}. Not every vertex of P defines a facet in P^{abl}. The origin is a trivial exceptional vertex, but there may be further exceptional vertices. We call a vertex v *dominated*, if there is another vertex w such that $v \leq w$. Now a vertex of P defines a facet of P^* if and only if it is not dominated.

Part II

Representations of Planar Graphs

Chapter 3

Planar graphs and polytopes

3.1 Planar graphs

A graph $G = (V, E)$ is *planar*, if it can be drawn in the plane so that its edges are continuous curves and they intersect only in their endpoints.

Theorem 3.1.1 (Kuratowski's Theorem) *A graph G is embedable in the plane if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$.*

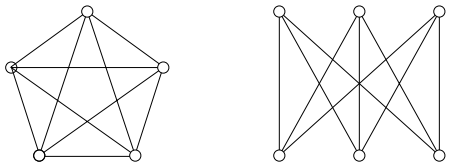

Figure 3.1: The two Kuratowski graphs.

Among planar graphs, 3-connected planar graphs are especially important.

A cycle C in a graph G is called *non-separating*, it it has no chords, and the removal of its nodes does not disconnect the graph.

Proposition 3.1.2 *Let G be a 3-connected planar graph, and C a cycle in G. Then C is non-separating if and only if it bounds a face.*

Corollary 3.1.3 *Every 3-connected planar graph has and essentially unique embedding in the plane.*

The following characterization of 3-connected planar graphs was proved by Tutte [129]:

Theorem 3.1.4 *Let G be a 3-connected graph. Then every edge of G is contained in at least two non-separating cycles. G is planar if and only if ever edge is contained in exactly two non-separating cycles.*

3.2 Straight line representation and 3-polytopes

Theorem 3.2.1 (Fáry–Wagner Theorem) *Every planar graph can be drawn with straight edges.*

Let P be a convex 3-polytope. The vertices and edges of P form a graph G_P, which we call the *skeleton* of P.

Proposition 3.2.2 *The skeleton of every 3-polytope is a 3-connected planar graph.*

We describe the simple proof, because this is our first example of how a geometric representation can be used to derive a purely graph-theoretic property, namely 3-connectivity.

Proof. Let F be any facet of P, and let x be a point that is outside P but very close to F; more precisely, assume that the plane Σ of F separates x from P, but for every other facet F', x is on the same side of the plane of F' as P. Let us project the skeleton of P from x to the plane Σ. Then we get an embedding of G_P in the plane.

To see that G_P is 3-connected, it suffices to show that for any four nodes a, b, c, d there is a path from a to b which avoids c and d.

If a, b, c, d are not coplanar, then let Π be a plane that separates $\{a, b\}$ from $\{c, d\}$; then we can connect a and b by a polygon consisting of edges of P that stays on the same side of Π as a and b, and so avoids c and d.

If a, b, c, d are coplanar, let Π be a plane that contains them. One of the open halfspaces bounded by Π contains at least one vertex of P. We can then connect a and b by a polygon consisting of edges of P that stays on this side of Π (except for its endpoints a and b), and so avoids c and d. □

The converse of this last proposition is an important and much more difficult theorem, proved by Steinitz [123]:

Theorem 3.2.3 (Steinitz's Theorem) *A simple graph is isomorphic to the skeleton of a 3-polytope if and only if it is 3-connected and planar.*

We don't prove Steinitz's Theorem here; constructions of representations by polytopes with special properties will follow from the material in chapters 4, 5.2 and 8 See also [109].

The construction of the planar embedding of G_P in the proof of Proposition 3.2.2 gives an embedding with straight edges. Therefore Steinitz's Theorem also proves the Fáry–Wagner theorem, at least for 3-connected graphs. It is easy to see that the general case can be reduced to this by adding new edges so as to make the graph 3-connected (see exercise 3.1.

Finally, we note that the Steinitz representation is also related to planar duality.

Proposition 3.2.4 *Let P be a convex polytope with the origin in its interior, and let P^* be its polar. Then the skeletons G_P are G_{P^*} are dual planar graphs.*

Exercise 3.1 Let G be a simple planar graph. Prove that you can add edges to G so that you make it 3-connected while keeping it planar.

Chapter 4

Rubber bands, cables, bars and struts

4.1 Rubber band representation

Let $G = (V, E)$ be a connected graph and $\emptyset \neq S \subseteq V$. Fix an integer $d \geq 1$ and a map $W : S \to \mathbb{R}^d$. We extend this to a map of all nodes of G into $\mathbb{R}^d$ (a geometric representation of G) as follows.

First, let's give an informal description. Replace the edges by ideal rubber bands (satisfying Hooke's Law). Think of the nodes in S as nailed to their given position (node $i \in S$ to $W_i \in \mathbb{R}^d$), but let the other nodes settle in equilibrium. We'll see that this equilibrium position is uniquely determined. We call it the *rubber band representation* of G in $\mathbb{R}^d$ extending W.

To be precise, let $x_i \in \mathbb{R}^d$ be the position of node $i \in V$. The *energy* of this representation is defined as

$$\mathcal{E}(x) = \sum_{ij \in E} |x_i - x_j|^2$$

We want to find the representation with minimum energy, subject to the boundary conditions:

$$\min\{\mathcal{E}(x) : \ x_i = W_i \text{ for all } i \in S\}.$$

Lemma 4.1.1 *if $W \neq \emptyset$, then the function $\mathcal{E}(x)$ is strictly convex.*

Proof.

□

It is trivial that if any of the x_i tends to infinity, then $\mathcal{E}(x)$ tends to infinity. With Lemma 4.1.1 this implies that the representation with minimum energy is uniquely determined. If $i \in V \setminus S$, then at the minimum point, the partial derivative of $\mathcal{E}(x)$ with respect to any coordinate of x must be 0. This means that for every $i \in V \setminus S$,

$$\sum_{j \in N(i)} (x_i - x_j) = 0. \tag{4.1}$$

This we can rewrite as

$$x_i = \frac{1}{d_i} \sum_{j \in N(i)} x_j. \tag{4.2}$$

This equation means that *every free node is in the center of gravity of its neighbors.* Equation (4.1) also has a nice meaning: the rubber band connecting i and j pulls i with force $x_j - x_i$, so (4.1) states that the forces acting on i sum to 0 (as they should at the equilibrium).

It will be convenient to extend this construction to the case when the edges of G have arbitrary positive weights. Let w_{ij} denote the weight of the edge ij. We then define the energy function of a representation $i \mapsto x_i$ by

$$\mathcal{E}(x) = \sum_{ij \in E} w_{ij} |x_i - x_j|^2.$$

The simple arguments above remain valid: there is a unique optimum, and for the optimal representation every $i \in V \setminus S$ satisfies

$$\sum_{j \in N(i)} w_{ij}(x_i - x_j) = 0. \tag{4.3}$$

This we can rewrite as

$$x_i = \frac{1}{\sum_{j \in N(i)} w_{ij}} \sum_{j \in N(i)} w_{ij} x_j. \tag{4.4}$$

Thus x_i is no longer in the center of gravity of its neighbors, but it is still a convex combination of them with positive coefficients. In other words, it is in the relative interior of the convex hull of its neighbors.

4.1.1 How to draw a graph?

The rubber band method was first analyzed by Tutte [129]. In this classical paper he describes how to use "rubber bands" to draw a 3-connected planar graph with straight edges and convex faces.

Let $G = (V, E)$ be a 3-connected planar graph, and let F be any face of it. Let C be the cycle bounding C. Let us map the nodes of C on the vertices of a convex polygon P in the plane, in the same cyclic order. Let $i \mapsto v_i$ be the rubber band representation of G in the plane with extending this map. We also draw the edges of G as straight line segments connecting the appropriate endpoints. We call this mapping the *rubber band representation of G with outer face C.*

By the above, we know that each node not on C is positioned at the center of gravity of its neighbors. Tutte's main result about this embedding is the following:

Theorem 4.1.2 *If G is a simple 3-connected planar graph, then its rubber band representation gives an embedding of G in the plane.*

Proof. The key to the proof is the following claim.

Claim 1. *Let ℓ be a line intersecting the polygon P, and let U be the set of nodes of G that are mapped onto a given side of ℓ. Then U induces a connected subgraph of G.*

Clearly the nodes of C in U form a path P. Let $a \in S \setminus V(C)$, then v_a is in the center of gravity of its neighbors, and so it has a neighbor a_1 such that v_{a_1} is on the same side of ℓ as a, but father from ℓ. Similarly, we find a neighbor a_2 of a_1 such that v_{a_2} is on the same side of ℓ as v_{a_1}, but farther, etc. This way we get a path Q in G that connects a to P, such that $V(Q) \subseteq S$. This proves the claim.

Now turning to the proof of theorem 4.1.2, we start with excluding a possible degeneracy. Call a node *degenerate* if there is a line such that the node and all its neighbors get positioned on this line.

Claim 2. *No degenerate nodes exist.*

Suppose that there are degenerate nodes; then there is a line ℓ which contains a node and all its neighbors. Fix this line, and consider the subgraph induced by all such nodes, and let H be a connected component of this subgraph (H may be a single node). Let S be the set of neighbors of H (outside H). Then $|S| \geq 3$ by 3-connectivity.

Let U_1 and U_2 be the sets of nodes of G on the two sides of ℓ. We claim that each node a in S is connected to both U_1 and U_2. By the definition of S, a is positioned on ℓ, but it has a neighbor that is not on ℓ, and so it has a neighbor in $U_1 \cup U_2$. If $a \notin V(C)$, then a is the center of gravity of its neighbors, and so it cannot happen that it has a neighbor on one side of ℓ but not the other. If $a \in V(C)$, then its two neighbors along C are on different sides of ℓ.

Now $V(H)$ induces a connected graph by definition, and U_1 and U_2 induce connected subgraphs by Claim 1. So we can contract these sets to single nodes. These three nodes will be adjacent to all nodes in S. So G can be contracted to $K_{3,3}$, which is a contradiction since it is planar. This proves Claim 2.

Claim 3. *Let ab be an edge that is not an edge of C, and let F_1 and F_2 be the two faces incident with ab. Then all other nodes of F_1 are mapped on one side of the line ℓ through v_a and v_b, and all other nodes of F_2 are mapped on the other side.*

Suppose not, then F_1 has a node c and F_2 has a node d such that v_c and v_d are both on (say) the positive side of ℓ, or on ℓ itself. In the latter case, they have a neighbor on the positive side of ℓ, by Claim 2. So by Claim 1, there is a path P connecting c and d whose internal nodes are positioned on the positive side of ℓ. Similarly, there is a path P' connecting a and b whose internal nodes are positioned on the negative side of ℓ. Thus P and P' are disjoint. But look at the planar embedding: the edge ab, together with P', forms a Jordan curve that separates b and d, so P cannot exist.

Claim 4. *The boundary of every face F is mapped onto a convex polygon P_F.*

This is immediate from Claim 3, since the line of an edge never intersects the interior of the face.

Claim 5. *The interiors of the polygons P_F (where F is a bounded face) are disjoint.*

Let x be a point inside P, we want to show that it is covered by one P_F only. Clearly we may assume that x not on the image of any edge. Draw a line through x that does not go through the image any node, and see how many times its points are covered by interiors of such polygons. As we enter P, this number is clearly 1. Claim 2 says that as the line crosses an edge, this number does not change. So x is covered exactly once.

Now the proof is essentially finished. Suppose that the images of two edges have a common point. Then two of the faces incident with them would have a common interior point, which is a contradiction except if these faces are the same, and the two edges are consecutive edges of this face. □

Before going on, let's analyze this proof a little. The key step, namely Claim 1, is very similar to a fact that we have seen before, namely Corollary 2.2.3. Let us call a geometric representation

of a graph *section-connected*, if for every open halfspace, the subgraph induced by those nodes that are mapped into this halfspace is connected (or empty). So the skeleton of a polytope, as a representation of itself, is section-connected; and so is the rubber-band representation of a planar graph. Note that the proof of Claim 1 did not make use of the planarity of G; in fact, the same proof gives:

Lemma 4.1.3 *Let G be a connected graph, and let w be a geometric representation of an induced subgraph H of G (in any dimension). If w is section-connected, then its rubber-band extension to G is also section-connected.*

4.1.2 How to lift a graph?

An old construction of Cremona–Maxwell can be used to "lift" Tutte's rubber band representation to a Steinitz representation.

Theorem 4.1.4 *Let $G = (V, E)$ be a 3-connected planar graph, and let T be a triangular face of G. Let $(v_i : i \in V)$ be a rubber band representation of G obtained by nailing T to any triangle in the plane. Then there is a function $\eta : V \to \mathbb{R}$ such that $\eta(i) = 0$ for $i \in V(T)$, $\eta(i) > 0$ for $i \in V \setminus V(t)$, and the mapping*

$$i \mapsto u_i = \begin{pmatrix} v_i \\ \eta(i) \end{pmatrix}$$

is a Steinitz representation of G.

Before starting with the proof, we need a little preparation to deal with edges on the boundary triangle. Recall that we can think of

$$F_{ij} = v_i - v_j$$

as the force with which the edge ij pulls its endpoint j. Equilibrium means that for every internal node j,

$$\sum_{i \in N(j)} F_{ij} = 0. \tag{4.5}$$

This does not hold for the external nodes, since those are nailed. but we can modify the definition of F_{ij} along the three boundary edges so that (4.5) will hold for all nodes.

(This is natural by physical intuition: let us replace the outer edges by rigid bars, and remove the nails. The whole structure will remain in equilibrium, so appropriate forces must act in the edges ab, bc and ac to keep balance. To translate this to mathematics, we have to work a little.)

Let

$$F_a = \sum_{i \in N(a) \setminus \{b,c\}} (v_i - v_a)$$

be the total force with which the internal edges pull a, and let F_b and F_c be defined similarly. If we sum (4.5) over all internal nodes, then edges connecting two internal nodes cancel, and we get the equation

$$F_a + F_b + F_c = 0. \tag{4.6}$$

To get a second equation, take the vectorial product of (4.5) with v_i, and sum over all internal nodes i. Since

$$v_i \times F_{ij} + v_j \times F_{ji} = (v_i - v_j) \times F_{ij} = (v_i - v_j) \times (v_i - v_j) = 0,$$

the internal edges cancel again, and we get

$$v_a \times F_a + v_b \times F_b + v_c \times F_c = 0. \tag{4.7}$$

(Equation (4.6) expresses, in physical terms, that the total pull of the internal rubber bands on the outer frame is 0, so it will not accelerate in any direction. Equation (4.7) says that the total torque is 0, so the frame will not start rotating.)

Draw a line ℓ_a parallel to F_a through x_a and a line ℓ_b parallel to F_b through v_b. Since F_a and F_b point inside the corresponding corners of the triangle T, these two lines intersect inside the triangle. We may choose this point of intersection as the origin. Then $v_a \parallel F_a$ and $v_b \parallel F_b$, and so $v_a \times F_a = v_b \times F_b = 0$. Thus (4.7) implies that $v_c \times F_c = 0$, and so $v_c \parallel F_c$.

Write $F_i = -\lambda_i v_i$ for $i = a, b, c$ (so that $\lambda_i > 0$). Define

$$F_{ij} = \frac{\lambda_i \lambda_j}{\lambda_a + \lambda_b + \lambda_c}(v_j - v_i)$$

for $i, j \in \{a, b, c\}$. We claim that with this definition, (4.5) will be satisfied for *all* nodes. We only have to check this e.g. for $i = a$. Then

$$\begin{aligned}\sum_{i \in N(a)} F_{ia} = F_a + F_{ba} + F_{ca} &= -\lambda_a v_a + \frac{\lambda_a \lambda_b}{\lambda_a + \lambda_b + \lambda_c}(v_a - v_b) + \frac{\lambda_a \lambda_c}{\lambda_a + \lambda_b + \lambda_c}(v_a - v_c) \\ &= \frac{\lambda_a}{\lambda_a + \lambda_b + \lambda_c}\Big(-(\lambda_a + \lambda_b + \lambda_c)v_a + \lambda_b(v_a - v_b) + \lambda_c(v_a - v_c)\Big) \\ &= -\lambda_a v_a - \lambda_b v_b - \lambda_c v_c = F_a + F_b + F_c = 0.\end{aligned}$$

Thus we have (4.5) for all nodes j. Now we are ready to prove theorem 4.1.4.

Proof. Imagine that we have the proper lifting. Let's call the third coordinate direction "vertical". For each face F, let g_F be a normal vector. Since no face is parallel to a vertical line, we can normalize g_F so that its third coordinate is 1. Clearly for each face F, g_F will be an outer normal, except for $F = T$, when g_F is an inner normal.

Write $g_F = \binom{h_F}{1}$. Let ij be any edge of G, and let F_1 and F_2 be the two faces incident with ij. Then both g_{F_1} and g_{F_2} are orthogonal to the edge $u_i u_j$ of the polytope, and therefore so is their difference. Since

$$(g_{F_1} - g_{F_2})^{\mathsf{T}}(u_i - u_j) = \left(\begin{pmatrix} h_{F_1} - h_{F_2} \\ 0 \end{pmatrix}\right)^{\mathsf{T}} \left(\begin{pmatrix} v_i - v_j \\ \eta(i) - \eta_j \end{pmatrix}\right) = (h_{F_1} - h_{F_2})^{\mathsf{T}}(v_i - v_j),$$

we get

$$(h_{F_1} - h_{F_2})^{\mathsf{T}}(v_i - v_j) = 0. \tag{4.8}$$

We also have

$$h_T = 0,$$

since the facet T is not lifted.

Using that not only $g_{F_1} - g_{F_2}$, but also g_{F_1} is orthogonal to the edge v_iv_j, we get from

$$g_{F_1}^{\mathsf{T}}(u_i - u_j) = h_{F_1}^{\mathsf{T}}(v_i - v_j) + (\eta_i - \eta_j)$$

that

$$\eta_i - \eta_j = -h_{F_1}^{\mathsf{T}}(v_i - v_j). \tag{4.9}$$

This discussion allows us to explain the plan of the proof: given the Tutte representation, we first reconstruct the vectors h_i so that all equations (4.8) are satisfied, then using these, we reconstruct the function η so that equations (4.9) are satisfied. It will not be hard to verify then that we get a Steinitz representation.

Let R denote the counterclockwise rotation in the plane by 90°. We claim that we can replace (4.8) by the stronger equation

$$h_{F_1} - h_{F_2} = RF_{ij} \tag{4.10}$$

and still have a solution. Starting with $h_F = 0$, and moving from face to adjacent face, this equation will determine the value of h_F for every face. What we have to show is that we don't run into contradiction, i.e., if we get to the same face F in two different ways, then we get the same vector h_F. This is equivalent to saying that if we walk around a closed cycle of faces, then the total change in the vector h_F is zero. We can think of this closed cycle as a Jordan curve in the plane, that does not go through any nodes, and crosses every edge at most once. We want to show that the sum of RF_{ij} over all edges that it crosses is zero (where the order of the endpoints is determined so that the Jordan curve crosses the edge from right to left).

From (4.5) we have that

$$\sum_{i \in N(j)} RF_{ij} = 0.$$

Summing this over all nodes j for which x_j lies in the interior of the Jordan curve, the terms corresponding to edges with both endpoints inside cancel (since $F_{ij} + F_{ji} = 0$), and we get that the sum is 0 for the edges crossing the Jordan curve. This proves that we can define the vectors h_F.

Second, we construct numbers η_i satisfying (4.9) by a similar argument. We set $\eta_i = 0$ if i is an external node. Equation (4.9) tells us what the value at one endpoint of an edge must be, if we have it for the other endpoint.

Again, the main step is to prove that we don't get a contradiction when coming back to a value we already defined.

The first concern is that (4.9) gives two conditions for each, depending on which face incident with it we choose. But if F_1 and F_2 are two faces incident with the edge ij, then

$$h_{F_1}^{\mathsf{T}}(v_i - v_j) - h_{F_1}^{\mathsf{T}}(v_i - v_j) = (h_{F_1} - h_{F_2})^{\mathsf{T}}(v_i - v_j) = (RF_{ij})^{\mathsf{T}}(v_i - v_j) = 0,$$

since F_{ij} is parallel to $v_i - v_j$ and so RF_{ij} is orthogonal to it. Thus the two conditions on the difference $\eta_i - \eta_j$ are the same; in other words,

$$\chi_{ij} = -h_F^{\mathsf{T}}(v_i - v_j)$$

depends only on the edge ij.

Now consider a cycle C in G, and (for reference) orient it counterclockwise. We want to show that the total change of η_i we prescribed along this cycle is 0. For every face F with boundary cycle ∂F, we have

$$\sum_{ij \in E(\partial F)} \chi_{ij} = \sum_{ij \in E(\partial F)} h_F^{\mathsf{T}}(v_i - v_j) = 0.$$

Summing this over all faces inside C the contribution of every edge cancels except for the edges on C. This proves that

$$\sum_{ij \in E(C)} \chi_{ij} = 0$$

as claimed.

Now define $u_i = \binom{v_i}{\eta_i}$ for every node i and $g_F = \binom{h_F}{1}$ for every face F. It remains to prove that $i \mapsto u_i$ maps the nodes of G onto the vertices of a convex polytope, so that edges go to edges and faces go to faces. We start with observing that if F is a face and ij is an edge of F, then

$$g_F^{\mathsf{T}} v_i - g_F^{\mathsf{T}} v_j = h_F^{\mathsf{T}}(v_i - v_j) + (\eta_i - \eta_j) = 0,$$

and hence there is a scalar α_F so that all nodes of F are mapped onto the hyperplane $g_F^{\mathsf{T}} x = \alpha_F$. We know that the image of F under $i \mapsto v_i$ is a convex polygon, and so the same follows for the map $i \mapsto u_i$.

To conclude, it suffices to prove that if ij is any edge, then the two convex polygons obtained as images of faces incident with ij "bend" in the right way; more exactly, let F_1 and F_2 be the two faces incident with ij, and let Q_{F_1} and Q_{F_2} be two two corresponding convex polygons. We claim that Q_{F_2} lies on the same side of the plane $g_{F_1}^{\mathsf{T}} x = \alpha_{F_1}$ as the bottom face. Let x be any point of the polygon Q_{F_2} not on the edge $u_i u_j$. We want to show that $g_{F_1}^{\mathsf{T}} x < \alpha_{F_1}$. Indeed,

$$g_{F_1}^{\mathsf{T}} x - \alpha_{F_1} = g_{F_1}^{\mathsf{T}} x - g_{F_1}^{\mathsf{T}} u_i = g_{F_1}^{\mathsf{T}}(x - u_i) = (g_{F_1} - g_{F_2})^{\mathsf{T}}(x - u_i)$$

(since both x and u_i lie on the plane $g_{F_2}^{\mathsf{T}} x = \alpha_{F_2}$),

$$= \begin{pmatrix} h_{F_1} - h_{F_2} \\ 0 \end{pmatrix}^{\mathsf{T}} (x - u_i) = (h_{F_1} - h_{F_2})^{\mathsf{T}}(x' - v_i)$$

(where x' is the projection of x onto the first two coordinates)

$$= (RF_{ij})^{\mathsf{T}}(x' - v_i) < 0$$

(since x' lies on the right hand side of the edge $v_i v_j$). This completes the proof. □

Theorem 4.1.4 proves Steinitz's theorem in the case when the graph has a triangular face. We are also home if the dual graph has a triangular face; then we can represent the dual graph as the skeleton of a 3-polytope, choose the origin in the interior of this polytope, and consider its polar; this will represent the original graph.

So the proof of Steinitz's theorem is complete, if we prove the following simple fact:

Lemma 4.1.5 *Let G be a 3-connected simple planar graph. Then either G or its complement has a triangular face.*

Proof. If G^* has no triangular face, then every node in G has degree at least 4, and so

$$|E(G)| \geq 2|V(G)|.$$

If G has no triangular face, then similarly

$$|E(G^*)| \geq 2|V(G^*)|.$$

Adding up these two inequalities and using that $|E(G)| = |E(G^*)|$ and $|V(G)| + |V(G^*)| = |E(G)| + 2$ by Euler's theorem, we get

$$2|E(G)| \geq 2|V(G)| + 2|V(G^*)| = 2|E(G)| + 4,$$

a contradiction. □

4.1.3 Rubber bands and connectivity

The idea of rubber bands can be related to graph connectivity, and can be used to give a test for k-connectivity of a graph.

Let $G = (V, E)$ be a graph and $S \subset V$, a fixed set of its nodes. A *convex representation of G in dimension d* (relative to S) is an embedding of V to $\mathbb{R}^d$ such that every node in $V \setminus S$ is in the convex hull of its neighbors. The representation is *in general position* if any $d + 1$ representing points are affine independent. The following fact was proved by Linial, Lovász and Wigderson [84].

Theorem 4.1.6 *A graph G has a convex representation (with respect to S) in general position in $\mathbb{R}^d$ if and only if no node of G can be separated from S by fewer than $d + 1$ nodes.*

4.1.4 Rubber bands and random walks

Hitting times can be expressed by a rubber band structure.

4.2 Rigidity of bar-and-joint structures

Tensegrity frameworks, stresses.

4.2.1 Cauchy's Theorem

Theorem 4.2.1 (Cauchy's theorem) *No 3-polytope carries a stress.*

Lemma 4.2.2 *If the edges of a planar map without multiple edges are 2-colored with red and blue, there is always a node where the red edges (and the blue edges) are consecutive.*

Exercise 4.1 Let S be a finite set of points in the plane, not all on a line. Color these points red and blue. Prove that there is a line which goes through at least two points in S and all whose points have the same color.

4.2.2 Generic rigidity

4.2.3 Stability

Chapter 5

Representing graphs by touching domains

5.1 Square tiling representation

Every planar triangulation of a square can be represented by a square tiling of a rectangle (Schramm [119]). Proof by minimizing the "energy" of a weighting of the nodes of a planar graph.

Aside: blocking polyhedra, energy.

Remark: every representation by touching homothetical copies of a centrally symmetric convex domain gives a straight line embedding (Figure 5.1).

5.2 Coin representation

We prove Koebe's important theorem on representing a planar graph by touching circles [77], and its extension to a polytopal representation, the Cage Theorem. Extensions by Andreev [5] and Thurston [128] to circles meeting at other angles will be described.

Nice applications of the Cage Theorem: a simple proof of the Planar Separator Theorem by Miller and Thurston [100], a bound on the eigenvalue gap of planar graphs by Spielman and Teng [122], and a bound on the cover time of planar graphs by Jonasson and Schramm [73].

5.3 The Cage Theorem

Theorem 5.3.1 *Every 3-connected planar graph is isomorphic to the 1-skeleton of a convex 3-polytope such that every edge of the polytope touches a given sphere.*

This is equivalent to a simultaneous representation of a 3-connected planar graph and of its dual by touching circles, so that circles representing an adjacent face-node pair are orthogonal. To be precise:

Theorem 5.3.2 *Let G be a 3-connected planar graph. Then one can assign to each node i a circle C_i and to each face a circle D_j on the sphere, so that for every edge ij, bordering faces a and b, the following holds: the circles C_i and C_j are tangent at a point p; the circles D_a and*

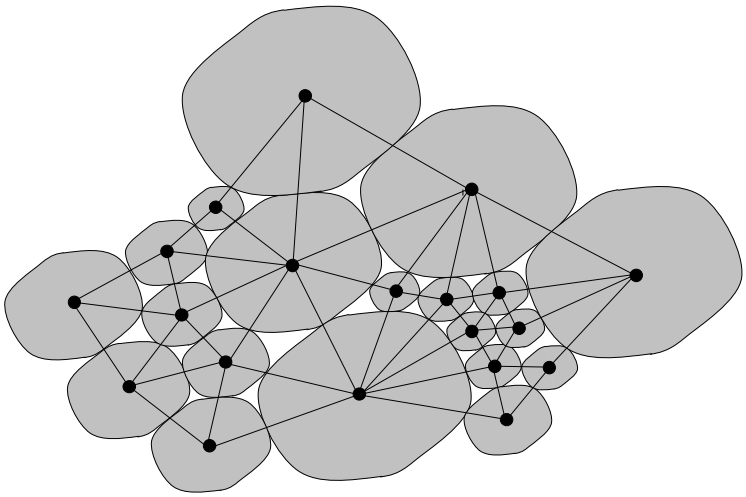

Figure 5.1: Straight line embedding of a planar graph from touching convex figures

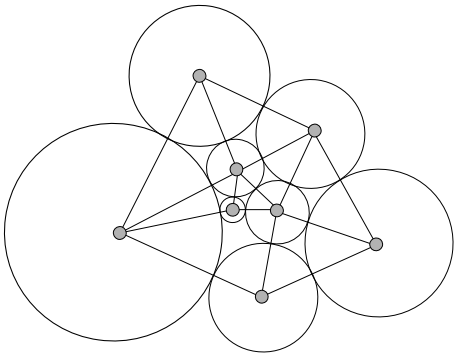

Figure 5.2: The coin representation of a planar graph

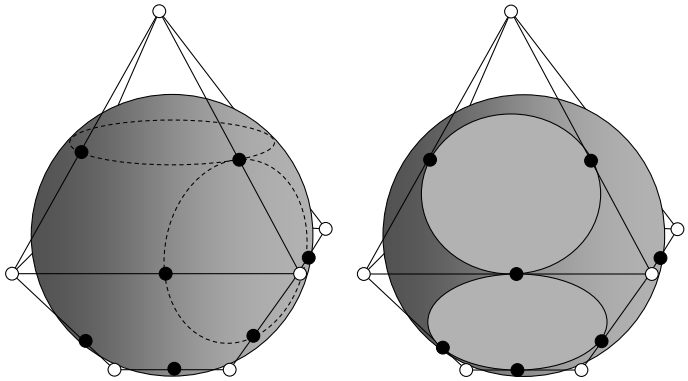

Figure 5.3:

D_b are tangent at the same point p; and the circles D_a and D_b intersect the circles C_i and C_b at this point.

Proof. Proof by Colin de Verdière. We fix a triangle as the outer face. For $i \in V$, let $NF(i)$ denote the set of faces containing i, and for $j \in \mathcal{F}$, let $NV(j)$ denote the set of nodes contained in j.

First, an auxiliary construction. Consider a Tutte rubber band embedding of the graph, with the outside triangle fixed to a regular triangle. For $i \in V$ and $j \in NF(i)$, let a_{ij} denote the angle of the polygon j at the vertex i. These numbers obviously have the following properties:

$$0 < a_{ij} < \pi, \tag{5.1}$$

$$\sum_{i \in N(j)} a_{ij} = 2(d_j - 2)\pi \tag{5.2}$$

for every bounded face j,

$$\sum_{j \in NF(i)} a_{ij} = 2\pi \tag{5.3}$$

for every node i not on the boundary, and

$$\sum_{j \in NF(i)} a_{ij} = \frac{\pi}{3} \tag{5.4}$$

for the three nodes i on the boundary.

Define

$$\phi(x) = 2 \int_{\infty}^{x} \arctan(e^t)\, dt.$$

Clearly ϕ is monotone increasing, convex, and

$$\phi(x) = \max\{0, \pi x\} + O(1). \tag{5.5}$$

Let $x \in \mathbb{R}^V$, $y \in \mathbb{R}^{\mathcal{F}}$, and consider

$$F(x,y) = \sum_{i,j:\, j \in N(i)} \Big(\phi(y_j - x_i) - a_{ij}(y_j - x_i)\Big).$$

Claim. If $|(x,y)| \to \infty$ while (say) $x_1 = 0$, then $F(x,y) \to \infty$.

We need to fix one of the x_i, since if we add the came value to each x_i and y_j, then the value of F does not change.

To prove the claim, we use (5.5):

$$\begin{aligned} F(x,y) &= \sum_{i,j:\, j \in N(i)} \Big(\phi(y_j - x_i) - a_{ij}(y_j - x_i)\Big) \\ &= \sum_{i,j:\, j \in N(i)} \Big(\max\{0, \pi(y_j - x_i)\} - a_{ij}(y_j - x_i)\Big) + O(1) \\ &= \sum_{i,j:\, j \in N(i)} \Big(\max\{-a_{ij}(y_j - x_i), (\pi - a_{ij})(y_j - x_i)\}\Big) + O(1). \end{aligned}$$

Since $-a_{ij}$ is negative but $\pi - a_{ij}$ is positive, each term here is non-negative, and a given term tends to infinity if $|x_i - y_j| \to \infty$. If x_1 remains 0 but $|(x,y)| \to infty$, then at least one difference $|x_i - y_j|$ must tend to infinity. This proves the Claim.

It follows from this Claim that F has a minimum at some point (x,y). Let i be an internal node, then

$$\frac{\partial}{\partial x_i} F(x,y) = -\sum_{j \in NF(i)} \phi'(y_j - x_i) + \sum_{j \in NF(i)} a_{ij}$$

$$= -2 \sum_{j \in NF(i)} \arccos(e^{y_j - x_i}) + 2\pi$$

(using (5.3)), and so

$$2 \sum_{j \in NF(i)} \arccos(e^{y_j - x_i}) = 2\pi. \tag{5.6}$$

It follows by a similar computation that

$$2 \sum_{j \in NF(i)} \arccos(e^{y_j - x_i}) = \frac{\pi}{3} \tag{5.7}$$

for the three boundary nodes i, and

$$2 \sum_{i \in NV(j)} \arccos(e^{y_j - x_i}) = (d_j - 2)\pi(i)$$

for every bounded face j. We can rewrite this condition as

$$2 \sum_{i \in NV(j)} \arccos(e^{x_i - y_j}) = 2\pi(i), \tag{5.8}$$

since $\arccos(e^{x_i - y_j}) + \arccos(e^{y_j - x_i}) = \pi/2$.

Now for every $i \in V$ and $j \in NF(i)$, we make two right triangles with sides e^{x_i} and e^{y_j}. From these triangles for a fixed j, we can make a convex polygon, using (5.8). These convex polygons will tile a regular triangle, by (5.6) and (5.7). □

Chapter 6

Harmonic and analytic functions on graphs

6.1 Harmonic functions

Let $G = (V, E)$ be a connected graph and $S \subseteq V$ (the orientation is not relevant right now). A function $\pi : V \to \mathbb{R}$ is called a "harmonic function with set of poles S" if

$$\frac{1}{d_v} \sum_{u \in N(v)} \pi(u) = \pi(v) \qquad \forall v \notin S.$$

It is trivial that every non-constant harmonic function has at least two poles (its minimum and maximum). For any two nodes $a, b \in V$ there is are harmonic functions with these poles. Such a harmonic function is uniquely determined up to scaling by a real number and translating by a constant. There are various natural ways to normalize; we'll somewhat arbitrarily decide on the following one:

$$\frac{1}{d_v} \sum_{u \in N(v)} (\pi(u) - \pi_v) = \begin{cases} 1, & \text{if } v = b, \\ -1, & \text{if } v = a, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

and

$$\sum_u \pi(u) = 0. \tag{6.2}$$

We denote this function by $\pi_{a,b}$ (if we want to express that it depends on a and b); if there is an edge e with $h(e) = b$ and $t(e) = a$, then we also denote $\pi_{a,b}$ by π_e. Expression (6.1) is equivalent to saying that $f_e = \pi(h(e)) - \pi(t(e))$ is a flow from a to b.

6.1.1 Harmonic functions from random walks, electrical networks, and rubber bands

Harmonic functions play an important role in the study of random walks: after all, the averaging in the definition can be interpreted as expectation after one move. They also come up in the

theory of electrical networks, and also in statics. This provides a connection between these fields, which can be exploited. In particular, various methods and results from the theory of electricity and statics, often motivated by physics, can be applied to provide results about random walks.

We start with describing three constructions of harmonic functions, one in each field mentioned.

Example 6.1.1 Let $\pi(v)$ denote the probability that a random walk starting at node v hits s before it hits t. Clearly, π is a harmonic function with poles s and t. We have $\pi(s) = 1$ and $\pi(t) = 0$.

More generally, if we have a set $S \subseteq V$ and a function $\pi_0 : S \to \mathbb{R}$, then we define $\pi(v)$ for $v \in V \setminus S$ as the expectation of $\pi_0(s)$, where s is the (random) node where a random walk starting at v first hits S. Then $\pi(v)$ is a harmonic function with pole set S. Moreover, $\pi(s) = \pi_0(s)$ for all $s \in S$.

Example 6.1.2 Consider the graph G as an electrical network, where each edge represents a unit resistance. Assume that an electric current is flowing through G, entering at s and leaving at t. Let $\pi(v)$ be the voltage of node v. Then π is a harmonic function with poles s and t.

Example 6.1.3 Consider the edges of the graph G as ideal springs with unit Hooke constant (i.e., it takes h units of force to stretch them to length h). Let us nail down nodes s and t to points 1 and 0 on the real line, and let the graph find its equilibrium. The energy is a positive definite quadratic form of the positions of the nodes, and so there is a unique minimizing position, which is the equilibrium. Clearly all nodes will lie on the segment between 0 and 1, and the positions of the nodes define a harmonic function with poles s and t.

More generally, if we have a set $S \subseteq V$ and we fix the positions of the nodes in S (in any dimension), and let the remaining nodes find their equilibrium, then every coordinate function is harmonic with pole set S.

A consequence of the uniqueness property is that the harmonic functions constructed (for the case $|S| = 2$) in examples 6.1.1, 6.1.2 and 6.1.3 are the same. As an application of this idea, we show the following interesting connections (see Nash-Williams [104], Chandra at al. [21]). Considering the graph G as an electrical network, let R_{st} denote the effective resistance between nodes s and t. Considering the graph G as a spring structure in equilibrium, with two nodes s and t nailed down at 1 and 0, let F_{ab} denote the force pulling the nails. Doing a random walk on the graph, let $\kappa(a, b)$ denote the commute time between nodes a and b (i.e., the expected time it takes to start at a, walk until you first hit b, and then walk until you first hit a again).

Theorem 6.1.4

$$\pi_{ab}(b) - \pi_{ab}(a) = R_{ab} = \frac{1}{F_{ab} = \frac{\kappa_{ab}}{2m}}.$$

Proof. By example 6.1.2, $\pi_{ab}(v)$ is the voltage of v if we push a unit current through G from a to b (conditions 6.2 is irrelevant here). So the effective resistance is $R_{ab} = \pi_{ab}(b) - \pi_{ab}(a)$.

The second equality is similarly easily derived from Hooke's Law.

Finally, Example 6.1.1 says that for an appropriate A and B, $A\pi_{ab}(u) + B$ is the probability that a random walk starting at u visits a before b. Checking this for $u = s$ and $u = t$, we get that $A = 1/(\pi_{ab}(b) - \pi_{ab}(a))$ and $B = -\pi_{ab}(u)/(\pi_{ab}(b) - \pi_{ab}(a))$. Hence $p_0 = \frac{1}{d_a} \sum_{u \in \Gamma(a)} (A(\pi_{ab}(u) - \pi_{ab}(a))$ is the probability that a random walk starting at a hits b before returning to t.

Let T be the first time when a random walk starting at a returns to a and S, the first time when it returns to a after visiting b. We know that $E(T) = 2m/d_a$ and by definition, $E(S) = \kappa(a, b)$. Clearly $T \leq S$ and the probability of $T = S$ is exactly p_0. This implies that $E(S - T) = (1 - p_0)E(S)$, since if $T < S$, then after the first T steps, we have to walk from a until we reach b and then return to b. Hence

$$p_0 = \frac{E(T)}{E(S)} = \frac{2m}{d_a \kappa(a, d)}.$$

□

Using the "topological formulas" from the theory of electrical networks for the resistance, we get a further characterization of these quantities:

Corollary 6.1.5 *Let G' denote the graph obtained from G by identifying a and b, and let $\mathcal{T}(G)$ denote the number of spanning trees of G. Then*

$$R_{ab} = \frac{\mathcal{T}(G)}{\mathcal{T}(G')}.$$

6.1.2 Harmonic and equitable flows

Let π be a function on a graph $G = (V, E)$, with pole set S. Let uv be any edge, and suppose that $\pi(u) \geq \pi(v)$. Orient the edge from u to v, and send a flow of

$$f_{uv} = \pi(u) - \pi(v) \tag{6.3}$$

from u to v. Then

Proposition 6.1.6 *The function f satisfies the flow condition at a node i if and only if π is harmonic at i.*

Indeed, if π is harmonic at i, then

$$\sum_j f(ij) = \sum_j (\pi(j) - \pi(i)) = \sum_j \pi(j) - d_i \pi(i) = 0;$$

the same computation also gives the converse.

Not every flow can be obtained from a harmonic function: for example, a non-zero circulation (a flow without sources and sinks) would correspond to a non-constant harmonic function with no poles, which cannot exist. In fact, the the flow obtained by (6.3) is *rotation-free* for every cycle C,

$$\sum_{ij} f(ij) = 0,$$

where the summation extends over all edges of C, oriented in the same direction around C.

6.1.3 Computing harmonic functions

Let χ_a denote the function which is 1 on a and 0 everywhere else. In terms of the Laplacian L of the graph, the equations (6.1) can be written as

$$L = \chi_b - \chi_a.$$

The matrix L is not quite invertible, but it has a one-dimensional nullspace spanned by the vector $\mathbf{1} = (1, \dots, 1)^\mathsf{T}$, and so it determines π up to adding the same scalar to every entry. We assumed in (6.2) that $\mathbf{1}^\mathsf{T}\pi = 0$. If $J \in \mathbb{R}^{V \times V}$ denotes the all-1 matrix, then

$$(L + J)\pi = L\pi = \chi_b - \chi_a,$$

and so we can express π as

$$\pi = (L + J)^{-1}(\chi_b - \chi_a). \tag{6.4}$$

6.2 Analytic functions

6.2.1 Circulations and homology

Let S be a closed compact surface, and consider a *map* on S, i.e., a graph $G = (V, E)$ embedded in S so that each face is a disc. We can describe the map as a triple $G = (V, E, \mathcal{F})$, where V is the set of nodes, E is the set of edges, and $\mathcal{F}$ is the set of faces of G. We fix a reference orientation of G; then each edge $e \in E$ has a *tail* $t(e) \in V$, a *head* $h(e) \in V$, a *right shore* $r(e) \in \mathcal{F}$, and a *left shore* $l(e) \in \mathcal{F}$.

The embedding of G defines a *dual map* G^*. Combinatorially, we can think of G^* as the triple $(\mathcal{F}, E, V)$, where the meaning of "node" and "face", "head" and "right shore", and "tail" and "left shore" is interchanged.

Let G be a finite graph with a reference orientation. For each node v, let $\delta v \in \mathbb{R}^E$ denote the coboundary of v:

$$(\delta v)_e = \begin{cases} 1, & \text{if } h(e) = v, \\ -1, & \text{if } t(e) = v, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $|\delta v|^2 = d_v$ is the degree of v.

For every face $F \in \mathcal{F}$, we denote by $\partial F \in \mathbb{R}^E$ the boundary of F:

$$(\partial F)_e = \begin{cases} 1, & \text{if } r(e) = F, \\ -1, & \text{if } l(e) = F, \\ 0, & \text{otherwise.} \end{cases}$$

Then $d_F = |\partial F|^2$ is the length of the cycle bounding F.

A vector $\phi \in \mathbb{R}^E$ is a *circulation* if

$$\phi \cdot \delta v = \sum_{e:\ h(e)=v} \phi(e) - \sum_{e:\ t(e)=v} \phi(e) = 0.$$

Each vector ∂F is a circulation; circulations that are linear combinations of vectors ∂F are called *null-homologous*. Two circulations ϕ and ϕ' are called *homologous* if $\phi - \phi'$ is null-homologous.

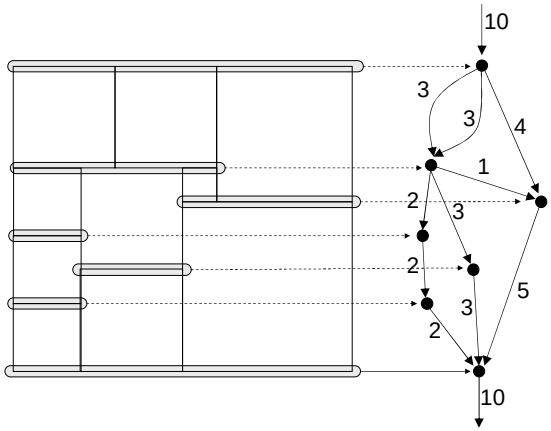

Figure 6.1: The Brooks–Smith–Stone–Tutte construction

Let ϕ be a circulation on G. We say that ϕ is *smooth* if for every face $F \in \mathcal{F}$, we have

$$\phi \cdot \partial F = 0.$$

This is equivalent to saying that ϕ is a circulation on the dual map G^*.

Smooth circulations can be considered as *discrete analytic functions* or more correctly *discrete holomorphic 1-forms*. These functions were introduced for the case of the square grid a long time ago [45, 32]. For the case of a general planar graph, the notion is implicit in [19]. For a detailed treatment see [99].

To explain the connection, let ϕ be a smooth circulation on a graph G embedded in a surface. Consider a planar piece of the surface. Then on the set $\mathcal{F}'$ of faces contained in this planar piece, we have a function $\sigma : \mathcal{F}' \to \mathbb{R}$ such that $\partial\sigma = \phi$, i.e., $\phi(e) = \sigma(r(e)) - \sigma(l(e))$ for every edge e. Similarly, we have a function $\pi : V' \to \mathbb{R}$ (where V' is the set of nodes in this planar piece), such that $\delta\pi = \phi$, i.e., $\phi(e) = \pi(t(e)) - \pi(h(e))$ for every edge e. We can think of π and σ as the real and imaginary parts of a (discrete) analytic function. The relation $\delta\pi = \rho\phi$ is then a discrete analogue of the Cauchy–Riemann equations.

Thus we have the two orthogonal linear subspaces: $\mathcal{A} \subseteq \mathbb{R}^E$ generated by the vectors δv ($v \in V$) and $\mathcal{B} \subseteq \mathbb{R}^E$ generated by the vectors ∂F ($F \in \mathcal{F}$). Vectors in $\mathcal{B}$ are 0-homologous circulations. The orthogonal complement $\mathcal{A}^\perp$ is the space of all circulations, and $\mathcal{B}^\perp$ is the space of circulations on the dual graph. The intersection $\mathcal{C} = \mathcal{A}^\perp \cap \mathcal{B}^\perp$ is the space of smooth circulations. So $\mathbb{R}^E = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$. From this picture we conclude the following.

Lemma 6.2.1 *Every circulation is homologous to a unique smooth circulation.*

It also follows that $\mathcal{C}$ is isomorphic to the first homology group of S (over the reals), and hence we get the following:

Theorem 6.2.2 *The dimension of the space $\mathcal{C}$ of smooth circulations is $2g$.*

6.2.2 Analytic functions from harmonic functions

We can use harmonic functions to give a more explicit description of smooth circulations in a special case. For any edge e of G, let η_e be the projection of χ_e onto $\mathcal{C}$.

Lemma 6.2.3 *Let $a, b \in E$ be two edges of G. Then $(\eta_a)_b$ is given by*

$$(\pi_b)_{h(a)} - (\pi_b)_{t(a)} + (\pi_b^*)_{r(a)} - (\pi_b^*)_{l(a)} + 1,$$

if $a = b$, and by

$$(\pi_b)_{h(a)} - (\pi_b)_{t(a)} + (\pi_b^*)_{r(a)} - (\pi_b^*)_{l(a)},$$

if $a \neq b$.

Proof. Let x_1, x_2 and x_3 be the projections of χ_b on the linear subspaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. The vector x_1 can be expressed as a linear combination of the vectors δv ($v \in V$), which means that there is a vector $y \in \mathbb{R}^V$ so that $x_1 = My$. Similarly, we can write $x_2 = Nz$. Together with $x = x_3$, these vectors satisfy the following system of linear equations:

$$\begin{cases} x + My + Nz = \chi_b \\ M^\mathsf{T} x = 0 \\ N^\mathsf{T} x = 0 \end{cases} \tag{6.5}$$

Multiplying the first m equations by the matrix M^T, and using the second equation and the fact that $M^\mathsf{T} N = 0$, we get

$$M^\mathsf{T} M y = M^\mathsf{T} \chi_b, \tag{6.6}$$

and similarly,

$$N^\mathsf{T} N z = N^\mathsf{T} \chi_b. \tag{6.7}$$

Here $M^\mathsf{T} M$ is the Laplacian of G and $N^\mathsf{T} N$ is the Laplacian of G^*, and so (6.6) implies that $y = \pi_b + c\mathbf{1}$ for some scalar c. Similarly, $z = \pi_b^* + c^*\mathbf{1}$ for some scalar c'. Thus

$$\begin{aligned} x &= \chi_b - M^\mathsf{T}(\pi_b + c\mathbf{1}) - N^\mathsf{T}(\pi_b^* + c^*\mathbf{1}) \\ &= \chi_b - M^\mathsf{T}\pi_b - N^\mathsf{T}\pi_b^*, \end{aligned}$$

which is just the formula in the lemma, written in matrix form. □

The case $a = b$ of the previous formula has the following formulation:

Corollary 6.2.4 *For an edge a of a map G, let R_a denote the effective resistance between the endpoints of a, and let R_a^* denote the effective resistance of the dual map between the endpoints of the edge dual to a. Then*

$$(\eta_a)_a = 1 - R_a - R_a^*.$$

6.2.3 Nondegeneracy properties of smooth circulations

We state and prove two key properties of smooth circulations: one, that the projection of a basis vector to the space of smooth circulations is non-zero, and two, that smooth circulations are spread out essentially over the whole graph in the sense that every connected piece of the graph where a non-zero smooth circulation vanishes can be isolated from the rest by a small number of points.

We start with a simple lemma about maps. For every face F, let a_F denote the number of times the orientation changes if we move along the the boundary of F. For every node v, let b_v denote the number of times the orientation changes in their cyclic order as they emanate from v.

Lemma 6.2.5 *Let $G = (V, E, \mathcal{F})$ be any digraph embedded on an orientable surface S of genus g. Then*

$$\sum_{F\in\mathcal{F}} (a_F - 2) + \sum_{v\in V} (b_v - 2) = 4g - 4.$$

Proof. Clearly

$$\sum_F a_F = \sum_v (d_v - b_v),$$

and so, using Euler's formula,

$$\sum_F a_F + \sum_v b_v = \sum_v d_v = 2m = 2n + 2f + 4g - 4.$$

Rearranging and dividing by 2, we get the equality in the lemma. □

If G is planar, then $R_a + R_a^* = 1$, a well known fact. For any other underlying surface, we have

$$(\eta_a)_a = \eta_a \cdot \chi_a = |eta_a|^2$$

(since η_a is a projection of χ_a), and so it follows that $R_a + R_a^* \leq 1$. It follows from theorem 6.2.6 below that strict inequality holds here.

If $g = 0$, then there is no nonzero smooth circulation by Theorem 6.2.2, and hence $\eta_e = 0$ for every edge e. But for $g > 0$ we have:

Theorem 6.2.6 *If $g > 0$, then $\eta_e \neq 0$ for every edge e.*

Proof. Suppose that $\eta_e = 0$ for some edge e. Then by Lemma 6.2.3, there are vectors $\pi = \pi(e) \in \mathbb{R}^V$ and $\pi^* = \pi^*(e) \in \mathbb{R}^F$ such that

$$\pi_{h(a)} - \pi_{t(a)} = \pi^*_{r(a)} - \pi^*_{l(a)} \tag{6.8}$$

for every edge $a \neq e$, but

$$\pi_{h(e)} - \pi_{t(e)} = 1 + \pi^*_{r(e)} - \pi^*_{l(a)}. \tag{6.9}$$

We define a convenient orientation of G. Let $E(G) = E_1 \cup E_2$, where E_1 consists of edges a with $\phi(h(a)) \not\phi(t(a))$, and E_2 is the rest. Every edge $a \in E_1$ is oriented so that $\pi h(a) > \pi t(a)$. Consider any connected component C of the subgraph formed by edges in E_2. Let $u_1, \ldots, u_k$ be the nodes of C that are incident with edges in E_1. Add a new node v to C and connect it to $u_1, \ldots, u_k$ to get a graph C'. Clearly C' is 2-connected, so it has an acyclic orientation such that every node is contained in a path from v to u_1. The corresponding orientation of C is acyclic and every has the property that it has no source or sink other than possibly $u_1, \ldots, u_k$.

Carrying this out for every connected component of G', we get an orientation of G. We claim this orientation is acyclic. Indeed, if we had a directed cycle, then walking around it π would never decrease, so it would have to stay constant. But then all edges of the cycle would belong to E_2, contradicting the way these edges were oriented.

We also claim this orientation has only one source and one sink. Indeed, if a node $v \neq h(e), t(e)$ is incident with an edge of E_1, then it has at least one edge of E_1 entering it and at

least one leaving it, by (6.1). If v is not incident with any edge of E_1, then it is an internal node of a component C, and so it is not a source or sink by the construction of the orientation of C.

Take the union of G and the dual graph G^*. This gives a graph H embedded in S. Clearly H inherits an orientation from G and from the corresponding orientation of G^*.

We are going to apply Lemma 6.2.5. Every face of H will $a_F = 2$ (this just follows from the way how the orientation of G^* was defined). Those nodes of H which arise as the intersection of an edge of G with an edge of G^* will have $b_v = 2$.

Consider a node v of G. If $v = h(a)$ then clearly all edges are directed toward v, so $b_{h(a)} = 0$. Similarly, we have $b_{t(v)} = 0$. We claim that $b_v = 2$ for every other node. Since obviously v is not a source or a sink, we have $b_v \geq 2$. Suppose that $b_v > 2$. Then we have for edges e_1, e_2, e_3, e_4 incident with v in this cyclic order, so that e_1 and e_2 form a corner of a face F, e_3 and e_4 form a corner of a face F', $h(e_1) = h(e_3) = v$ and $t(e_2) = t(e_3) = v$.

Consider π^* of the faces incident with v. We may assume that $\pi^*(F) \leq \pi^*(F')$. From the orientation of the edges e_1 and e_2 it follows that $\pi^*(F)$ is larger than π^* of its neighbors. Let $\mathcal{F}$ be the union of all faces F'' with $\pi^*(F'') \geq \pi^*(F)$. The boundary of $\mathcal{F}$ is an eulerian subgraph, and so it can be decomposed into edge-disjoint cycles $D_1, \ldots, D_t$. Since the boundary goes through v twice (once along e_1 and e_2, once along two other edges with the corner of F' on the left hand side), we have $t \geq 2$, and so one of these cycles, say D_1, does not contain e. But then by the definition of the orientation and by (6.8), D_1 is a directed cycle, which is a contradiction.

A similar argument shows that if v is a node corresponding to a face not incident with e, then $b_v = 0$; while if v comes from $r(e)$ or from $l(e)$, then $b_v = 2$.

So substituting in Lemma 6.2.5, only two terms on the left hand side will be non-zero, yielding $-4 = 4g - 4$, or $g = 0$. □

Corollary 6.2.7 *If $g > 0$, then for every edge e, $(\eta_e)_e \geq n^{-n} f^{-f}$.*

Indeed, combining with the remark after Corollary 6.2.4, we see that $(\eta_e)_e > 0$ if $g > 0$. But $(\eta_e)_e = 1 - R_e - R_e^*$ is a rational number, and it is easy to see that its denominator is not larger than $n^n f^f$.

Theorem 6.2.8 *Let G be a graph embedded in an orientable surface S of genus $g > 0$ so that all faces are discs. Let ϕ be a non-zero smooth circulation on G and let G' be the subgraph of G on which ϕ does not vanish. Suppose that ϕ vanishes on all edges incident with a connected subgraph U of G. Then U can be separated from G' by at most $4g - 3$ points.*

The assumption that the connectivity between U and the rest of the graph must be linear in g is sharp in the following sense. Suppose X is a connected induced subgraph of G separated from the rest of G by $\leq 2g$ nodes, and suppose (for simplicity) that X is embedded in a subset of S that is topologically a disc. Contract X to a single point x, and erase the resulting multiplicities of edges. We get a graph G' still embedded in S so that each face is a disc. Thus this graph has a $(2g)$-dimensional space of circulations, and hence there is a non-zero smooth circulation ψ vanishing on $2g - 1$ of the edges incident with x. Since this is a circulation, it must vanish on all the edges incident with x. Uncontracting X, and extending ψ with 0-s to the edges of X, it is not hard to check that we get a smooth circulation.

Proof. Let W be the connected component of $G \setminus V(G')$ containing U, and let Y denote the set of nodes in $V(G) \setminus V(W)$ adjacent to W.

Consider an edge e with $\phi(e) = 0$. If e is not a loop, then we can contract e and get a map on the same surface with a smooth flow on it. If $G - e$ is still a map, i.e., every face is a disc, then ϕ is a smooth flow on it. If $G - e$ is not a map, then both sides of e must be the same face.

So we can eliminate edges with $\phi(e) = 0$ unless $h(e) = t(e)$ and $r(e) = l(e)$ (we call these edges strange loops). In this latter case, we can change $\phi(e)$ to any non-zero value and still have a smooth flow.

Applying this reduction procedure, we may assume that $W = \{w\}$ consists of a single node, and the only edges with $\phi = 0$ are the edges between w and Y, or between two nodes of Y. We cannot try to contract edges between nodes in Y (we don't want to reduce the size of Y), but we can try to delete them; if this does not work, then every such edge must have $r(e) = l(e)$.

Also, if more than one edge remains between w and a node $y \in Y$, then each of them has $r(e) = l(e)$ (else, one of them could be deleted). Note that we may have some strange loops attached at w. Let D be the number of edges between w and Y.

Re-orient each edge with $\phi \neq 0$ in the direction of the flow ϕ, and orient the edges between w and Y alternatingly in an out from w. Orient the edges with $\phi = 0$ between two nodes of Y arbitrarily. We get a digraph G_1.

It is easy to check that G_1 has no sources or sinks, so $b_v \geq 2$ for every node v, and of course $b_w \geq |Y| - 1$. Furthermore, every face either has an edge with $\phi > 0$ on its boundary, or an edge with $r(e) = l(e)$. If a face has at least one edge with $\phi > 0$, then it cannot be bounded by a directed cycle, since ϕ would add up to a positive number on its boundary. If a face boundary goes through an edge with $r(e) = l(e)$, then it goes through it twice in different directions, so again it is not directed. So we have $a_F \geq 2$ for every face.

Substituting in Lemma 6.2.5, we get that $|Y| - 1 \leq 4g - 4$, or $|Y| \leq d_w \leq 4g - 3$. Since Y separates U from G', this proves the theorem. □

Part III

Representations in Higher Dimensions

Chapter 7

Orthogonal representations

We want to label the nodes of a graph by vectors in $\mathbb{R}^d$ so that nonadjacent nodes must be labeled by orthogonal vectors. Various parameters of such labelings are related to interesting graph properties.

7.1 Smallest cone and the theta function

The smallest angle of a rotational cone (in arbitrary dimension) which contains all vectors in an orthogonal representation of the graph gives rise to the theta-function of the graph [87]. Unlike the smallest dimension, this quantity is polynomial time computable and is closely related to the independence number and the chromatic number.

See [54, 76] for more detail.

7.2 Minimum dimension and connectivity

The minimum dimension in which such a labeling exists seems difficult to determine, but we get interesting results if we impose some "non-degeneracy" conditions. A result of Lovász, Saks, Schrijver [90] (see [4] for an application in quantum computing) finds an exact condition for this type of geometric representability.

Theorem 7.2.1 *A graph G with n nodes has a general position orthogonal representation in $\mathbb{R}^d$ if and only if it is (at least) $n - d$-connected.*

If we replace the condition that any d vectors are linearly independent by a weaker condition called the *Strong Arnold Property*, the smallest d will be closely related to the tree-width of the graph (Colin de Verdiére [27]; see next section).

7.3 Treewidth and monotone connectivity

Tree-width is a parameter related to connectivity, introduced by Robertson and Seymour [110] as an important element in their graph minor theory. Colin de Verdière defines the *tree-width* $\mathrm{tw}(G)$ of a graph G as the smallest r for which G embeds to the cartesian sum of the complete graph K_r and a tree. (This is not quite the same as the more standard notion of tree-width

introduced by Robertson and Seymour, but the difference is at most 1, as shown by van der Holst [63]).

A related parameter is the *monotone connectivity* $\kappa_{\mathrm{mon}}(G)$ of a graph G, defined as the maximum node-connectivity of minors of G. It is easy to see that

$$\kappa_{\mathrm{mon}}(G) \leq \mathrm{tw}(G).$$

Let d be the smallest dimension in which an orthogonal representation with the Strong Arnold Property exists, and define $a(G) = n - d$. The main advantage of this nondegeneracy condition is that it implies that $a(G)$ is *minor-monotone*, i.e., if G is a minor of G, then $a(H) \leq a(G)$.

Colin de Verdière showed that $a(G)$ is a lower bound on the tree-width of the graph. Combining with Theorem 7.2.1 and the minor-monotonicity of $a(G)$ we can also bound $a(G)$ from below. These bounds are summed up in the following theorem.

Theorem 7.3.1 *For every graph,*

$$\kappa_{\mathrm{mon}} \leq a(G) \leq \mathrm{tw}(G).$$

Colin de Verdière conjectured that equality holds in the upper bound. This was proved by Van der Holst [63] and Kotlov [78] for $a(G) \leq 2$, but in general it is false: it is not hard to see that the k-cube has $a(G) = O(2^{k/2})$ but $\mathrm{tw}(G) = \Theta(2^k)$. It is not known whether $\mathrm{tw}(G)$ can be larger than $a(G)^2$.

Chapter 8

The Colin de Verdière Number

This exciting graph parameter $\mu(G)$ can be defined (informally) as the multiplicity of the second largest eigenvalue of the adjacency matrix, where we can weight the edges and the diagonal entries to maximize this multiplicity [25], and we impose a non-degeneracy condition (for a survey, see [69]).

8.1 The definition

8.1.1 The Strong Arnold Property

Let $\mathbb{R}^{V^{[2]}}$ denote the space of all symmetric $V \times V$ matrices. Let $\mathcal{R}_k \subseteq \mathbb{R}^{V^{[2]}}$ be the manifold of symmetric $V \times V$ matrices with rank k (Figure 8.1).

Lemma 8.1.1 *Let $M \in \mathcal{R}^k$, and let S denote the matrix of the orthogonal projection onto the nullspace of M. Let $\mathcal{T}(M)$ and denote $\mathcal{N}(M)$ denote the tangent space and normal space of $\mathcal{R}_k$ at M, respectively.*

(a) *For every symmetric $V \times V$ matrix Y, the following are equivalent:*

(a1) $Y \in \mathcal{T}(M)$*;*

(a2) $SYS = 0$*;*

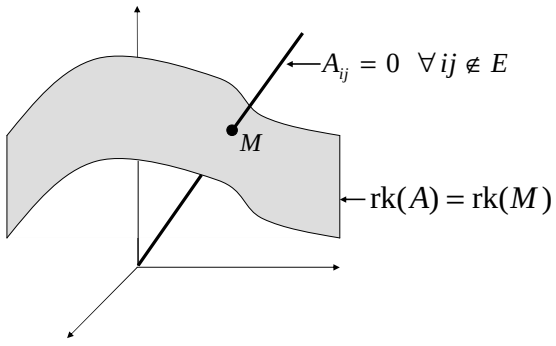

Figure 8.1: The Strong Arnold Hypothesis

(a3) *there exists a (not necessarily symmetric)* $V \times V$ *matrix* U *such that* $Y = MU + U^{\mathsf{T}}M$.

(b) *For every symmetric* $V \times V$ *matrix* X*, the following are equivalent:*

(b1) $X \in \mathcal{N}(M)$*;*

(b2) $MX = 0$*;*

(b3) $X = SXS$*;*

(b4) *there exists a symmetric* $V \times V$ *matrix* W *such that* $X = SWS$.

Proof. ((a1) $\Rightarrow$ (a2)). Let $Y \in \mathcal{T}(M)$. Then there is a one-parameter differentiable family $M(t)$ of symmetric matrices, defined in a neighborhood of $t = 0$, so that $M(t) \in \mathcal{R}_k$, $M(0) = M$ and $M'(0) = Y$. Let $S(t)$ denote the matrix of the orthogonal projection onto the nullspace of $M(t)$, so that $S(0) = S$.

By definition, we have $M(t)S(t) = 0$, and hence by differentiation, we get $M'(0)S(0) + M(0)S'(0) = 0$, or $YS + MS'(0) = 0$. Multiplying by S from the left, we get $SYS = 0$. So

((a2) $\Rightarrow$ (a3)). Using that $SYS = 0$, we can write

$$Y = \frac{1}{2}((I - S)Y(I + S) + (I + S)Y(I - S)).$$

Notice that $I - S$ is the orthogonal projection onto the range of M, and hence we can write $I - S = MV$ with some matrix V. By transposition, we have $I - S = V^{\mathsf{T}}M$. Then

$$Y = \frac{1}{2}MVY(I + S) + \frac{1}{2}(I + S)YV^{\mathsf{T}}M = MU + U^{\mathsf{T}}M,$$

where $U = \frac{1}{2}VY(I + S)$.

((a3) $\Rightarrow$ (a1)). Suppose that $Y = MU + U^{\mathsf{T}}M$. Consider the family

$$M(t) = (I + tU)^{\mathsf{T}}M(I + tU).$$

Clearly $\mathrm{rank}(M(t)) \leq \mathrm{rank}(M) \leq k$ and equality holds if $|t|$ is small enough. Furthermore, $M'(0) = MU + U^{\mathsf{T}}M = Y$. Hence $Y \in \mathcal{T}(M)$.

II. ((b1) $\Rightarrow$ (b2)). If $X \in \mathcal{N}(M)$, then $X \cdot Y = 0$ for every $Y \in TT(M)$, which by (a3) means that $X \cdot (MU + U^{\mathsf{T}}M) = 0$ for every U. This can be written as $\mathrm{Tr}(X(MU + U^{\mathsf{T}}M)) = 0$. But $\mathrm{Tr}(X(MU + U^{\mathsf{T}}M)) = \mathrm{Tr}(XMU) + \mathrm{Tr}(XU^{\mathsf{T}}M) = \mathrm{Tr}(XMU) + \mathrm{Tr}(MUX) = 2\mathrm{Tr}(XMU)$, and so it follows that $\mathrm{Tr}(XMU) = 0$ for every matrix U. This implies that $XM = 0$.

((b2) $\Rightarrow$ (b3)). If $MX = 0$, then by transposition $XM = 0$, and so the range of X is contained in the kernel of M, which is just the range of the projection S. So $XS = X$, and by transposition, $SX = X$, and so $SXS = SX = X$.

((b3) $\Rightarrow$ (b4)) is trivial.

((b4) $\Rightarrow$ (b1)). Suppose that $X = SWS$, then by (a2), we have for every $Y \in \mathcal{T}(M)$ we have $Y \cdot X = \mathrm{Tr}(YSWS) = \mathrm{Tr}(SYSW) = 0$. □

8.2 Basic properties

Theorem 8.2.1 *The Colin de Verdière number is minor-monotone.*

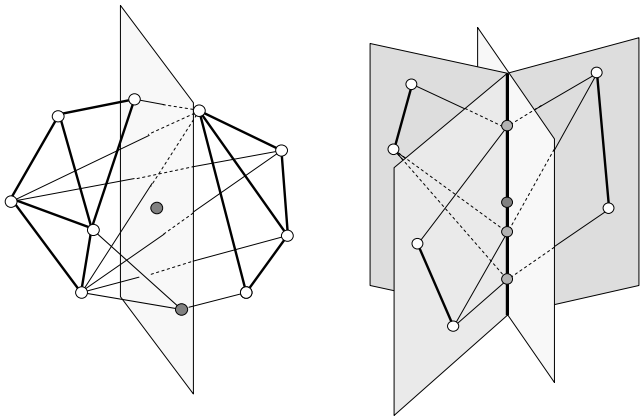

Figure 8.2: Two possibilities for the nullspace representation of a connected graph: for every hyperplane through the origin, either both sides are connected, or...

Theorem 8.2.2 *If $\mu(G) > 2$, then $\mu(G)$ is invariant under subdivision.*

The following result was proved by Bacher and Colin de Verdière [7].

Theorem 8.2.3 *If $\mu(G) > 3$, then $\mu(G)$ is invariant under $\Delta - Y$ transformation.*

8.3 Small values

Graphs with Colin de Verdière number up to 4 are characterized.

Theorem 8.3.1 *(a) $\mu(G) \leq 1$ if and only if G is a path;*

(b) $\mu(G) \leq 2$ if and only if G is outerplanar;

(c) $\mu(G) \leq 3$ if and only if G is planar;

(d) $\mu(G) \leq 4$ if and only if G is linklessly embedable.

Statements (a)–(c) were proved by Colin de Verdière [25]; an elementary proof is due to Hein van der Holst [62]); (d) is due to Lovász and Schrijver [91].

8.4 Nullspace representation

Every weighted adjacency matrix with a d-dimensional nullspace gives rise to an embedding of the graph in d-space. Van der Holst's Lemma has a nice geometric meaning in this context (Figure 8.2).

From the Colin de Verdière matrix (the optimal matrix in the definition of $\mu(G)$), we get a representation of the graph in $\mu(G)$-space.

For 3-connected planar graphs, the nullspace representation gives a Steinitz representation [92, 89]; in fact, Steinitz representations naturally correspond to Colin de Verdiére matrices.

8.5 Gram representation

Every Colin de Verdiére matrix of a graph gives rise to another geometric representation of the graph, this time in the $(n - \mu(G) - 1)$-dimensional space [79]. This is related to the Koebe–Andre'ev representation if the complement of the graph is a maximal planar graph.

8.6 The Van der Holst–Laurent–Schrijver parameter

Van der Holst's lemma motivates this related graph parameter, which is again related to planarity and other geometric representations [64].

Chapter 9

Graph independence to linear independence

9.1 Independence-critical graphs

As a useful tool in the study of graphs critical with respect to stability number, Lovász [86] considered vector representations with the property that *every set of nodes that covers all the edges, spans the space*. We call this *cover-preserving*. One can dualize this notion (see section 12), to get the following condition on vector representations: *every stable (independent) set of nodes is represented by linearly independent vectors*. Obviously, every orthogonal representation has this property.

In this case, the dimension problem is trivial: such a representation exists in dimension $\alpha(G)$ (the maximum number of independent nodes) and higher. But independence-preserving representations become interesting in conjunction with criticality: namely, assuming that deleting any edge, the representation does not remain independence-preserving. We refer to [86] for details.

Chapter 10

Metric embeddings

10.1 Embeddings of metric spaces

Besides prescribing edge-lengths, geometric representations of graphs with special requirements like (approximately) distance-preserving [83] and volume-preserving [44] embeddings have been studied. These are closely related to classical results of Johnson–Lindenstrauss [71] and Bourgain [13] on embedding finite metric spaces. See also the monograph [31].

10.2 Multicommodity flows and bandwidth

A fundamental result in the theory of multicommodity flows is the theorem of Leighton and Rao [82]. Stated informally (and in a larger generality, as proved by Linial, London and Rabinovich [83]), it says the following. Suppose that we have a multicommodity flow problem on a graph on n nodes. Obvious cut-conditions provide a system of necessary conditions for the problem to be feasible; but (unlike for the case of a single commodity), these conditions are not sufficient in general. The theorem asserts that *if the cut-conditions are satisfied, then relaxing the capacities by a factor of $O(\log n)$, the problem becomes feasible.*

The proof of Linial, London and Rabinovich depends on "distance-preserving" embeddings of graphs. Given a graph, we would like to embed it in a euclidean space so that the distances between nodes in the graph should be the same, or at least close to, the geometric distance of the representing vectors. It is not hard to see that one will necessarily have some distortion in non-trivial cases. For example, the "claw" $K_{1,3}$ cannot be embedded without distortion. So we are interested in two parameters: the dimension and the distortion. Often, the dimension problem is easy to handle, due to a fundamental lemma of Johnson and Lindenstrauss [71], which asserts that every n-point configuration in $\mathbb{R}^n$ can be embedded in $\mathbb{R}^d$ with $d = O(\log n)$ with arbitrarily small distortion.

This topic is closely related to the area of embedding metric spaces in each other, and we can only refer to the book of Deza and Laurent [31] for the combinatorial aspects of such embedding problems. However, we mention one line of development because of its rich graph-theoretic applications.

Bourgain [13] proved that every metric space with n elements can be embedded in an $O(\log n)$-dimensional euclidean space with $O(\log n)$ distortion; more precisely, he constructs an embedding in which the geometric distance is at most the original, and at least $1/\log n$ times the original. Matoušek [98] showed that for an expander graph this is best possible.

Linial, London and Rabinovitch [83] extended Bourgain's work in various directions. For our survey, the most important application they give is a proof and quite substantial extension of a fundamental result of Leighton and Rao [82] in the theory of multicommodity flows.

A very interesting new application of this construction was given by Feige [44]. In an algorithm that finds a polylogarithmic approximation of the bandwidth of a graph in polynomial time, he uses geometric representations of graphs in $\mathbb{R}^d$, with the following properties:

(a) the representation is contractive, i.e., the distance between the endpoints of any edge is at most 1;

(b) the representation is volume respecting, meaning that every "small" set of nodes spans a simplex with almost as large volume as possible.

Obviously, (b) needs explanation. Consider any set S of k nodes. Let T be a shortest spanning tree on S (with respect to the graph distance $d(.,.)$; T is not a subgraph of G). It is easy to see from (a) that the volume of the simplex spanned by S is at most

$$\mathrm{TREEVOL}(S) = \frac{1}{k!} \prod_{uv \in E(T)} d(u,v).$$

Formulating the result for one reasonable choice of parameters, Feige constructs a contractive geometric representation in dimension $d = O((\log n)^3)$, such that for each set S of at most $\log n$ nodes, the volume of the simplex spanned by S is at least $\mathrm{TREEVOL}(S)/(\log n)^{2|S|}$.

The ordering of the nodes which approximates the bandwidth is now obtained through a random projection of the representation to the line, in a fashion similar to the Goemans–Williamson algorithm above.

Part IV

General issues

Chapter 11

Semidefinite optimization

Semidefinite relaxations of various combinatorial optimization problems lead to interesting and useful geometric representations (Goemans and Williamson [51]).

Chapter 12

Is there a theory of geometric representations

Is there a way to fit these geometric representations in a single theory? Perhaps not, considering the variety of possibilities how the graph structure can be reflected in the geometry. Nevertheless, there are some general ideas that can be pointed out.

12.1 Graph structure and geometry

Let us start with a summary of graph representations connecting the graph structure to geometry.

Perhaps the most natural idea is the original definition of dimension by Erdős, Harary and Tutte [40]: require that *all edges have the same length*, say 1. It seems that, at the same time, this notion is rather difficult, and very simple questions like Hadwiger's problem on the chromatic number of unit distance graphs in the plane remain unsolved. We seem to get a closer connection with the graph structure if we require that the edges correspond to the minimum or maximum distance. At least questions about the density and chromatic number of graphs representable this way have more complete answers, as we have seen.

Minimum-distance representations are, of course, equivalent to representing a graph by touching circles (or balls) of the same size, and so they are also connected to Koebe's touching disk representation. In fact, the Cage Theorem can be formulated as 3-dimensional minimum distance representability in a certain projective metric.

Considering other metrics brings quite a few other representations into this framework. Orthogonal representations can be viewed as maximum distance representations in elliptic spaces: if instead of the vectors, we consider the lines of these vectors, then the maximum angular distance between two lines is 90 degrees. Orthogonal representations have been very useful in the study of independent sets and cliques, chromatic number (section 7.1), Shannon capacity [86], connectivity and treewidth (section 7.2).

Putting restrictions on *inner products* instead of distances of adjacent pairs leads to quite similar questions, which can often be reduced to each other.

A third, natural but not too well understood version is to consider *threshold representations*: let the edges correspond to pairs of points closer than a certain prescribed threshold. Bounds on the number of edges and on the degrees were proved by Frankl and Maehara [47]. In the setting of inner products, [46], [107] and [108] studied such questions.

Another version is obtained when we want to preserve the graph distance of all pairs (not just adjacent pairs). Constructing approximations has been the main tool in section 10.2.

Some of the constructions above were weaker: only the ordered structure of the space was used. *Convex representations* were related to graph connectivity (section 7.2); *connected-halfspace representations*, to topological properties (Chapter 8).

Finally, one might use only the linear structure of $\mathbb{R}^n$. Cover-preserving and independence-preserving representations, discussed in section 9, have this property.

12.2 Non-degeneracy

A common theme in connection with various representations is that imposing non-degeneracy conditions on the representation often makes it easier to analyze and therefore more useful (basically, by eliminating the possibility of numerical coincidence). There are at least 3 types of non-degeneracy conditions; we illustrate the different possibilities by formulating them in the case of unit distance representations in $\mathbb{R}^d$. All three are easily extended to other kinds of representations.

The most natural non-degeneracy condition is *faithfulness*: we want to represent the graph so that adjacent nodes *and only those* are at unit distance. This is usually not strong enough. *General position* means that no $d+1$ of the points are contained in a hyperplane.

Perhaps the deepest non-degeneracy notion is the following. Write the condition of unit distance representation as a system of algebraic equations:

$$\|u_i - u_j\|^2 = 1 \qquad (ij \in E).$$

We have nd unknowns (the coordinates of the u_i). Each of these equations defines a hypersurface in $\mathbb{R}^{nd}$, and a representation corresponds to a point where these hypersurfaces intersect. Now we say that this representation has the *Strong Arnold Property* if the hypersurfaces intersect transversally, i.e., their normal vectors at this point are linearly independent. This condition means that the intersection point is not just accidental, but is forced by some more fundamental structure; for example, if the representation has the Strong Arnold Property, and we change by a small amount each constant 1 on the right hand sides of the defining equations, we get another solvable system.

12.3 Duality

The following notion of *duality* is known under many aliases: dual chain group in matroid theory, dual code in coding theory, Gale diagram in the theory of hyperplane arrangements, etc. Let $u_1, \dots, u_n \in \mathbb{R}^d$. Write down these vectors as column vectors, and let L be the row space of the resulting matrix. Pick any basis in the orthogonal complement $L^\perp$ of L, write them down as row vectors, and let $v_1, \dots, v_n \in \mathbb{R}^{n-d}$ be the columns of the resulting matrix. One of the main properties of this construction is that a set of the u_i forms a basis of $\mathbb{R}^d$ if and only if the complementary set of the v_i forms a basis of $\mathbb{R}^{n-d}$.

We can carry out this construction for any vector representation of a graph G, to get a *dual vector representation*. In some cases, this gives interesting constructions; for example, from cover-preserving representations we get independence-preserving representations. But note that (at least in the definition above) the dual is only determined up to an affine transformation; for geometric representations with metric properties (which is the majority), dualization does

not seem to make sense. Yet it seem that in some cases more than the basic linear structure is dualized, and we don't have a general explanation for this. Let us briefly mention two examples.

In [53], a duality for orthogonal representations of a graph and its complement has been described. One of the consequences is that every graph G has an orthogonal representation whose dual (in the sense described above) becomes an orthogonal representation of the complementary graph $\overline{G}$, if an appropriate single row is added. This result is connected to the duality theory of semidefinite programming.

In [79], it was pointed out that there seems to be a duality between the Colin de Verdère numbers of planar graphs and their complements. Again (up to a single row) the nullspace representation and the Gram representation derived from a Colin de Verdère matrix of a graph are dual to each other; but while the Gram representation has strong metric properties, it is unclear how to impose those on the nullspace representation.

12.4 Algorithmic applications

To represent a graph geometrically is a natural goal in itself, but in addition it is an important tool in the study of various graph properties, including their algorithmic aspects. There are several levels of this interplay between algorithms and geometry.

— Often the aim is to find a way to represent a graph in "good" way. We refer to Kuratowski's characterization of planar graphs, its more recent extensions most notably by Robertson and Seymour, and to Steinitz's theorem representing 3-connected planar graphs by 3-dimensional polyhedra. Many difficult algorithmic problems in connection with these representations have been studied.

— In other cases, graphs come together with a geometric representation, and the issue is to test certain properties, or compute some parameters, that connect the combinatorial and geometric structure. A typical question in this class is rigidity of bar-and-joint frameworks, an area whose study goes back to the work of Cauchy and Maxwell.

— Most interesting are the cases when a good geometric representation of a graph leads to algorithmic solutions of purely graph-theoretic questions that, at least on the surface, do not seem to have anything to do with geometry. Our discussions contained several examples of this (but the list will be far from complete): graph connectivity, graph coloring, finding maximum cliques in perfect graphs, giving capacity bounds in information theory, approximating the maximum cut and the bandwidth, planarity, linkless embedability, and rigidity of frameworks.

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