Towards Capturing Order-Independent P

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Restrict attention to the complexity of computing individual bits of the output, i.e., decision problems. How hard is it to check if input has property $S$? How rich a language do we need to express property $S$? There is a computable isomorphism between these two approaches.
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How rich a language do we need to express property $S$?

There is a computable isomorphism between these two approaches.
Think of the Input as a Finite Logical Structure

\[ H = (\{a, b, c\}, \leq, E^H, R^H, G^H, B^H) \]

Colored Graph

\[ H \]

\[ \begin{array}{ccc}
  a & \rightarrow & b \\
  \downarrow & & \downarrow \\
  c & \rightarrow & b
\end{array} \]
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Colored \[ E^H = \{(a, b), (b, a), (b, c), (c, b), (c, a), (a, c)\} \]

Graph

\[ H \]

![Graph diagram]

\[ H \]
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Graph
\[ R^H = \{a\} \]
\[ G^H = \{b\} \]
\[ B^H = \{c\} \]
Think of the Input as a Finite Logical Structure

$$H = (\{a, b, c\}, \leq, E^H, R^H, G^H, B^H)$$

$$\leq^H = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$$

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$$E^H = \{(a, b), (b, a), (b, c), (c, b), (c, a), (a, c)\}$$

Graph

$$R^H = \{a\}$$

$$G^H = \{b\}$$

$$B^H = \{c\}$$
First-Order Logic

input symbols:   \( E, R, Y, B, \ldots \)
variables:       \( x, y, z, \ldots \)
boolean connectives: \( \land, \lor, \neg \)
quantifiers:     \( \forall, \exists \)
numeric symbols: \( =, \leq, +, \times, \text{min}, \text{max} \)

\[ \alpha \equiv \forall x \exists y \ E(x, y) \]
\[ \beta \equiv \forall xy (\neg E(x, x) \land (E(x, y) \rightarrow E(y, x))) \]
\[ \gamma \equiv \forall x ((\forall y \ x \leq y) \rightarrow R(x)) \]
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\]

In this setting, with the structure of interest being the finite input, FO is a weak complexity class.
First-Order Logic

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In this setting, with the structure of interest being the finite input, FO is a weak complexity class.

It is easy to test if input, $H$, satisfies $\alpha$ ($H \vDash \alpha$).
First-Order Logic

\[ H \quad a \leq b \leq c \]

\[ G \quad 1 \leq 2 \leq 3 \]

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\[ H \equiv a \leq b \leq c \]

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\[ H \models \alpha \land \beta \land \gamma \]

\[ G \models \alpha \land \beta \land \neg \gamma \]

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\( \alpha \) and \( \beta \) are order independent; \( \gamma \) is order dependent
Fagin's Theorem: $\text{NP} = \text{SO}$

$\Phi_{3\text{color}} \equiv \exists R^1 G^1 B^1 \forall x y ((R(x) \lor G(x) \lor B(x)) \land (E(x, y) \rightarrow \neg(R(x) \land R(y)) \land \neg(G(x) \land G(y)) \land \neg(B(x) \land B(y))))$
Fagin’s Theorem: \( \text{NP} = \text{SO} \exists \)

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\]
Inductive Definitions and Least Fixed Point

\[ \text{REACH} = \{ G, s, t \mid s \rightarrow^* t \} \]
\[ \text{REACH} = \{ G, s, t \mid s \xrightarrow{*} t \} \quad \text{REACH} \not\in \text{FO} \]
Inductive Definitions and Least Fixed Point

\[ E^*(x, y) \overset{\text{def}}{=} x = y \lor E(x, y) \lor \exists z(E^*(x, z) \land E^*(z, y)) \]

\begin{align*}
\text{REACH} &= \{ G, s, t \mid s \xrightarrow{*} t \} \\
\text{REACH} &\notin \text{FO}
\end{align*}
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\[ E^*(x, y) \overset{\text{def}}{=} x = y \lor E(x, y) \lor \exists z (E^*(x, z) \land E^*(z, y)) \]

\[ \varphi_{tc}(R, x, y) \equiv x = y \lor E(x, y) \lor \exists z (R(x, z) \land R(z, y)) \]

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\[ \varphi^G_{tc} : \text{binRel}(G) \rightarrow \text{binRel}(G) \]

\text{monotone} \quad R \subseteq S \Rightarrow \varphi^G_{tc}(R) \subseteq \varphi^G_{tc}(S) \]

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\[ E^* = (\text{LFP}\varphi_{tc}) \]

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\[ \text{monotone} \quad R \subseteq S \implies \varphi^G_{tc}(R) \subseteq \varphi^G_{tc}(S) \]

\[ G \in \text{REACH} \iff G \models (\text{LFP} \varphi_{tc})(s, t) \quad E^* = (\text{LFP} \varphi_{tc}) \]

\[ \text{REACH} = \{ G, s, t \mid s \xrightarrow{*} t \} \quad \text{REACH} \notin \text{FO} \]
Thm. \[ P = \text{FO}(\text{LFP}) = \text{FO}[n^{O(1)}] \]

\text{FO}[n^{O(1)}] \text{ means for graphs with } n \text{ vertices, the formula } \varphi_n \text{ expressing the property has } n^{O(1)} \text{ quantifiers, but only a fixed number of requantified variables}, x_1, \ldots, x_k, \text{ i.e., } \varphi_n \in \mathcal{L}^k.
LFP is a Polynomial Iteration Operator

**Thm.** \[ P = \text{FO}(\text{LFP}) = \text{FO}[n^{O(1)}] \]

Graphs are completely general structures, i.e., any structure can be encoded as a graph.

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Theorem. $P = \text{FO}(\text{LFP}) = \text{FO}[n^{O(1)}]$

Graphs are completely general structures, i.e., any structure can be encoded as a graph. \textbf{Restrict to graphs.}

$\text{FO}[n^{O(1)}]$ means for graphs with $n$ vertices, the formula $\varphi_n$ expressing the property has $n^{O(1)}$ quantifiers, but only a fixed number of requantified variables, $x_1, \ldots, x_k$, i.e, $\varphi_n \in \mathcal{L}^k$. 
**Thm.** \( P = \text{FO}(\text{LFP}) = \text{FO}[n^{O(1)}] \)

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Above **Thm** requires ordering relation, \( \leq \).
Thm. \[ P = FO(LFP) = FO[n^{O(1)}] \]

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Necessary for encoding computation – inputs to computers are ordered.
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Unnatural for graphs – the ordering of the vertices is irrelevant.
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Graphs are completely general structures, i.e., any structure can be encoded as a graph. **Restrict to graphs.**

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Necessary for encoding computation – inputs to computers are ordered.

Unnatural for graphs – the ordering of the vertices is irrelevant.

**Wanted:** a language capturing Order-Independent P (**OIP**).
Want to Capture Order-Independent P (OIP)

$$\text{FO}(\text{LFP}) = \text{P}$$

$$\text{FO}(\text{wo}\leq)(\text{LFP}) \subseteq \text{OIP}$$
Want to Capture Order-Independent P (OIP)

\[ \text{FO} = \text{P} \]

\[ \text{FO}(\text{wo} \leq) \text{(LFP)} \subseteq \text{OIP} \]

\[ \text{EVEN} \overset{\text{def}}{=} \{ G \mid |V^G| \equiv 0 \pmod{2} \} \]
Want to Capture Order-Independent P (OIP)

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\[ \text{EVEN} \in \text{OIP} - \text{FO}(\text{wo} \leq) (\text{LFP}). \]

Thus, \[ \text{FO}(\text{wo} \leq) (\text{LFP}) \nsubseteq \text{OIP} \]
Want to Capture Order-Independent P (OIP)

\[ \text{FO}(\text{LFP}) = P \]

\[ \text{FO}(\text{wo} \leq)(\text{LFP}) \subseteq \text{OIP} \]

\[ \text{EVEN} \overset{\text{def}}{=} \{ G \mid |V^G| \equiv 0 \pmod{2} \} \]

\[ \text{EVEN} \in \text{OIP} - \text{FO}(\text{wo} \leq)(\text{LFP}). \]

Thus, \[ \text{FO}(\text{wo} \leq)(\text{LFP}) \not\subseteq \text{OIP} \]

How do we prove EVEN \( \not\in \text{FO}(\text{wo} \leq)(\text{LFP}) \)?
Ehrenfeucht-Fraïssé Game

$g^k_m(G, H)$  \hspace{1em} m moves, \hspace{1em} k pebbles, \hspace{1em} 2 players

Samson: show a difference.

Delilah: preserve isomorphism.

For all $m$, $D$ wins $g^2_m(G, H)$; but $S$ wins $g^3_3(G, H)$. 
Ehrenfeucht-Fraïssé Game

$g_m^k(G, H)$  \(m\) moves, \(k\) pebbles, 2 players

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Ehrenfeucht-Fraïssé Game

$G^k_m(G, H)$, $m$ moves, $k$ pebbles, 2 players

Samson: show a difference. Delilah: preserve isomorphism.
Ehrenfeucht-Fraïssé Game

\[ g^k_m(G, H) \quad m \text{ moves}, \quad k \text{ pebbles}, \quad 2 \text{ players} \]

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**Samson**: show a difference. **Delilah**: preserve isomorphism.

For all \(m\), \(D\) wins \(G^2_m(G, H)\);

\[
\begin{align*}
G & = \{a, b, c, d, e, f\} \\
H & = \{g, h, i, j, k, l\}
\end{align*}
\]
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![Diagram of graphs G and H]
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\[
\begin{array}{c}
\text{G} \\
\text{f} & \text{e} & \text{d} \\
\text{b} & \text{c} \\
\text{a} \\
\end{array}
\quad \quad
\begin{array}{c}
\text{H} \\
\text{g} & \text{h} & \text{i} \\
\text{l} & \text{k} & \text{j} \\
\text{x_1} & \text{x_2} & \text{x_3} \\
\end{array}
\]
Ehrenfeucht-Fraïssé Game

\[ G^k_m(G, H) \] \( m \) moves, \( k \) pebbles, 2 players

**Samson**: show a difference. **Delilah**: preserve isomorphism.

For all \( m \), \( D \) wins \( G^2_m(G, H) \); but \( S \) wins \( G^3_3(G, H) \).
Notation: \( G \sim_{m}^{k} H \) means that Delilah has a winning strategy for \( G_{m}^{k}(G, H) \).
Notation: $G \sim_{m}^{k} H$ means that Delilah has a winning strategy for $G^{k}_{m}(G, H)$.

Thm. D has a winning strategy on the $m$-move, $k$-pebble game on $G, H$ iff $G$ and $H$ agree on all formulas using $k$ variables and quantifier depth $m$.

$$G \sim_{m}^{k} H \iff G \equiv_{m}^{k} H$$
Thm. \textbf{EVEN} requires $n + 1$ variables without ordering. Thus \textbf{EVEN} $\not\in \text{FO(wo}\leq)(\text{LFP})$. 
Thm. **EVEN** requires \( n + 1 \) variables without ordering. Thus **EVEN** \( \notin \text{FO(wo} \leq \text{)(LFP)} \).

**proof:**

\[
\begin{align*}
G_{2m} & : \quad \cdots \quad \vdots \quad \cdots \quad H_{2m+1} \\
g_1 & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
g_2 & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
g_{2m} & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
h_1 & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
h_2 & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
h_{2m} & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
h_{2m+1} &
\end{align*}
\]
Thm. **EVEN** requires \( n + 1 \) variables without ordering. Thus \( \text{EVEN} \not\in \text{FO}(\text{wo}\leq)(\text{LFP}) \).

**proof:**

\[
\begin{array}{ccc}
  g_1 & x_1 & h_1 \\
g_2 \\
\vdots \\
g_{2m} \\
  G_{2m} \\
\end{array}
\quad
\begin{array}{ccc}
  h_1 \\
h_2 \\
\vdots \\
h_{2m} \\
  H_{2m+1} \\
\end{array}
\]
Thm. **EVEN** requires \( n + 1 \) variables without ordering. Thus \( \text{EVEN} \not\in \text{FO(wo\leq)}(\text{LFP}) \).

proof:

\[
\begin{align*}
g_1 & \quad x_1 \\
g_2 & \\
g_{2m} & \\
\vdots & \\
G_{2m} & \\
\end{align*}
\begin{align*}
h_1 & \quad x_1 \\
h_2 & \\
h_{2m} & \\
\vdots & \\
H_{2m+1} & \\
\end{align*}
\]

\[
\begin{align*}
&g_1 \\
&g_2 \\
&g_{2m} \\
&h_1 \\
&h_2 \\
&h_{2m} \\
&h_{2m+1} \\
\end{align*}
\]
Thm. \( \text{EVEN} \) requires \( n + 1 \) variables without ordering. Thus \( \text{EVEN} \notin \text{FO}(\text{wo} \leq) (\text{LFP}) \).

proof:

\[ g_1 \quad x_1 \]
\[ g_2 \]
\[ \vdots \]
\[ g_{2m} \]
\[ G_{2m} \]

\[ h_1 \quad x_1 \]
\[ h_2 \quad x_2 \]
\[ \vdots \]
\[ h_{2m} \]
\[ h_{2m+1} \]
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**Thm.**  
**EVEN** requires \( n + 1 \) variables without ordering. Thus **EVEN** \( \not\in \text{FO(wo}\leq\text{)(LFP)} \).

**proof:**

\[
\begin{array}{c}
g_1 & x_1 \\
g_2 & x_2 \\
\vdots & \\
g_{2m} & \\
\end{array}
\quad
\begin{array}{c}
h_1 & x_1 \\
h_2 & x_2 \\
\vdots & \\
h_{2m} & \\
\end{array}
\quad
\begin{array}{c}
G_{2m} \\
\vdots \\
g_{2m} & \\
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\quad
\begin{array}{c}
h_1 & x_1 \\
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**proof:**

\[
\begin{align*}
g_1 & \quad x_1 \\
g_2 & \quad x_2 \\
\vdots & \\
g_{2m} & \quad x_{2m} \\

\end{align*}
\quad
\begin{align*}
h_1 & \quad x_1 \\
h_2 & \quad x_2 \\
\vdots & \\
h_{2m} & \\

\end{align*}
\quad
\begin{align*}
G_{2m} & \\
\vdots & \\
H_{2m+1} & \\

\end{align*}
\quad
\begin{align*}
h_{2m} \\

\end{align*}
\quad
\begin{align*}
h_{2m+1} \\

\end{align*}
\]
Thm. **EVEN** requires $n + 1$ variables without ordering. Thus **EVEN** ∉ FO(wo≤)(LFP).

**proof:**

$$
\begin{align*}
G_{2m} & \vdash x_1 \\
g_1 & \vdash x_1 \\
g_2 & \vdash x_2 \\
\vdots & \\
g_{2m} & \vdash x_{2m} \\

H_{2m+1} & \vdash x_1 \\
h_1 & \vdash x_1 \\
h_2 & \vdash x_2 \\
\vdots & \\
h_{2m} & \vdash x_{2m} \\
h_{2m+1} & \\
\end{align*}
$$
**Thm.** \( \text{EVEN} \) requires \( n + 1 \) variables without ordering. Thus \( \text{EVEN} \not\in \text{FO}(\text{wo}\leq)(\text{LFP}) \).

**proof:**

\[
\begin{align*}
G_{2m} & \sim 2m H_{2m+1}
\end{align*}
\]
Two sorts: **Numbers**: \(\{0, 1, \ldots, n\}\), \(\leq\), Plus, Times and **Vertices**: \(\{v_1, \ldots, v_n\}\), \(E, C_1, C_2 \ldots\)
Add Counting to FO Logic

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Combine with counting terms:  \( \#x(\varphi(x)) \).
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Let \( C^k \overset{\text{def}}{=} \operatorname{FO}^k(\text{COUNT}) \); \( \text{FPC} \overset{\text{def}}{=} \operatorname{FO}(\text{LFP}, \text{COUNT}) \).
Add Counting to FO Logic

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\( \text{FO}(\text{wo} \leq)(\text{LFP}) \subsetneq \text{FPC} \subseteq \text{OIP} \)
Stable Coloring of Vertices

Start with a colored graph, and repeatedly color each vertex by how many neighbors it has of each color.
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Thm. Stable Coloring of Vertices = \( C_2 \) type. Round \( m \) of stable coloring is quantifier depth of \( C_2 \) formula.
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**Thm.** Stable Coloring of Vertices $= C^2$ type.

Round $m$ of stable coloring is quantifier depth of $C^2$ formula.
Thm. [Babai, Erdos, Selkow] With high probability, after four iterations of stable coloring, each vertex of a random graph has a unique color, i.e., the $C_4^2$-type of each vertex is unique.
The Good News: Upper Bounds

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Thus, for almost all graphs, there is a linear time algorithm to canonize the graph, i.e., sort the vertices by their $C^2$ type, so that two graphs are isomorphic iff their canonical forms are equal.
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Thus, Graph Isomorphism (GI) is linear time for random graphs.

In general the complexity of GI is unknown.

**Thm.** [Babai, 2015] $\text{GI} \in \text{DTIME}[n^{\log_7 n}]$. (Before this it was only known that $\text{GI} \in \text{DTIME}[n^{\sqrt{n}}].$)
**Def.** Language $\mathcal{L}$ characterizes a graph $G$ iff for all graphs $H$,

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Logics Characterizing Graphs

**Def.** Language $\mathcal{L}$ **characterizes** a graph $G$ iff for all graphs $H$,

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- $C^2$ characterizes almost all random graphs.
- $C^2$ characterizes all trees.
- $C^3$ characterizes all graphs of color class size 3.
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Cor. If \( C^k \) characterizes all graphs in a class of graphs \( \mathcal{G} \) that is closed under particularizing, then \( \mathcal{G} \) admits \( C^k \) canonization, and thus FPC captures \textbf{OIP} over \( \mathcal{G} \).

proof: Apply arbitrary FO(LFP) formula to the canonical form of the input graph.
Particularizing Means Uniquely Coloring Some Vertex
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Diagram:
- Vertex a is highlighted.
- Vertices b, c, d, e, f are connected in a triangle.
- Vertices g, h, i, j, k, l are connected in a hexagonal structure.
- There is a connection between a and g.
Is FPC Equal to OIP?

▶ Is FPC Equal to OIP?
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  Thus, GI would be in \( \text{DTIME}[n^4 \log n] \).

**Thm.** [CFI]  No!

A simple graph property (now called the CFI property) checkable in \( \text{DTIME}[n] \), requires \( v = \Omega(n) \) variables to express in \( C^v \). Thus, \( \text{CFI} \in \text{OIP} - \text{FPC} \).
Proof of CFI Thm

Each \( m_i \) adjacent to an even number of \( a_j \)'s.

Automorphisms of \( X \): switch an even number of (\( a_i b_i \)) pairs.

\begin{align*}
\text{Automorphism:} \\
(a_2 b_2) (a_3 b_3) (m_1 m_2) (m_3 m_4)
\end{align*}

\begin{align*}
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(a_1 b_1) (a_2 b_2) (m_1 m_4) (m_2 m_3)
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If we remove any $n$ vertices from $G_n$, it still has a connected component with more than $|V^{G_n}|/2$ vertices.
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Thus $X(G_n)$ has color class size 4.
$X(G_n)$: replace each vertex $v \in V^{G_n}$ by a copy of $X$ of $v$’s color, connecting $a$ to $a$ and $b$ to $b$. 
$\tilde{X}(G)$ is $X(G)$ with any one edge pair flipped.
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Prop. Let $X'(G_n)$ be $X(G_n)$ with some number, $m$, of the magenta edges flipped.

Then $X'(G_n) \cong X(G_n)$ iff $m$ is even and

\[ X'(G_n) \cong \tilde{X}(G_n) \text{ iff } m \text{ is odd.} \]
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proof: Using the automorphisms of $X$, we can move any two flips towards each other until they eliminate each other.
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$\tilde{X}(G_n)$
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Every one of the $m_i$’s is connected to an even number of $a_j$’s.
Every one of the $m_i$'s is connected to an odd number of $a_j$'s.
Def. CFI = \{(X'(G) \mid X'(G) \cong X(G)\} for G is connected, reg. deg. 3, cc(G) = 1.
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Prop. \( \text{CFI} \in \text{DTIME}[\eta] \).
The CFI Problem

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Prop.  \( \text{CFI} \in \text{DTIME}[n] \).

proof Use the ordering to label boundary pairs \( a_i, b_i \) when \( a_i \leq b_i \). Then count the number, \( m \), of flips of vertices and edges mod 2. \( X'(G) \in \text{CFI} \) iff \( m \) is even.  

.
$\tilde{X}(G_n)$
Thm. CFI ∈ OIP – FPC.
**Thm.** CFI $\in$ OIP $-$ FPC.

**proof** We show that $X(G_n) \equiv_{C^n} \tilde{X}(G_n)$. Counting doesn't help since $c_c(X(G_n)) = 4$. Suffices to show that $X(G_n) \sim \tilde{X}(G_n)$.

Initially no pebbles on the board, Samson places $x_1$ on $X(v)$ in one of the two graphs. Note that the largest connected component $C_1$ of $G - \{v\}$ includes over half the vertices of $G$. Delilah moves the flip into $C_1$. If she removes the flip, then the two graphs are isomorphic. Delilah answers according to this isomorphism. Inductively, after step $m$, Delilah has not yet lost, so there is an isomorphism from chosen points in $X(G_n)$ to chosen points in $\tilde{X}(G_n)$ which extends to an isomorphism of the whole graphs in which a flip in $\tilde{G}_n$ in $C_m$ has been removed.
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Thus **Delilah** never loses. □
Recap

We have shown that the linear-time CFI problem is in OIP – FPC.

Cor. \( \Omega(n) \) variables are needed to characterize graphs.
Martin Grohe has shown that many classes of graphs are characterized by $C^k$ for some $k$. This includes planer graphs, graphs of bounded genus, graphs of bounded tree width and culminating in

Thm. [Anderson, Dawar and Holm] Linear Programming is in FPC.
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Thus,

- FPC captures **OIP** on $\mathcal{G}$. Thus, for graphs from $\mathcal{G}$, graph isomorphism and canonization are in P.

- For $G, H \in \mathcal{G}$, $G \cong H$ iff $G \equiv_{C^k} H$. 
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**Thm.** [Anderson, Dawar and Holm] Linear Programming is in FPC.
Two other languages are candidates for capturing OIP:

- Choiceless Polynomial Time (CPT) [Blass and Gurevich]
  Compute using sets of sets of sets, etc., where instead of choosing the first vertex, we consider the set of all such choices, keeping the total size of all sets polynomial.

- Rank Logic [Dawar, Grohe, Holm, and Laubner]
  Compute the rank of matrices expressed in an unordered setting.

CFI is expressible in CPT and in Rank Logic, thus these are strict extensions of FPC.
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Going Beyond FPC

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What I want: more natural extension to **FPC** that adds group theory and characterizes graphs using $O(\log n)$ variables.